Ecole Doctorale Carnot-Pasteur

Proposition de sujet de thèse

Intitulé français du sujet de thèse proposé :

Orthogonalité totale et fonctions d'énumération de racines généralisées pour catégories de fusion groupe-théoriques

Intitulé en anglais du sujet de these proposé :

Total orthogonality and generalized root counting functions for group-theoretical fusion categories

Unité de recherche : IMB (UMR 5584, Université de Bourgogne & CNRS)

Nom, prénom et courriel du directeur (et co-directeur) de thèse :

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Domaine scientifique principal de la thèse :

Mathématiques, mathématique physique

Description du projet scientifique :

In most general terms, the project will concern the structure theory and numerical invariants of fusion categories (which are, to give incomplete axiomatics at least, semisimple tensor categories with duality).

I will start with a very specific problem, which can serve as an example but should not be viewed as the sole aim of the thesis; in particular it is quite possible that it will continue to escape a solution.

The representations of the Drinfeld double of a finite group are a particular example of a (modular, in particular braided) fusion category. The higher Frobenius-Schur indicators for the simple objects of a fusion category are numerical invariants closely linked to the applications of fusion categories in quantum topology. They are classical invariants in the particular case of representations of finite groups (where they are always integers). For the representations of the Drinfeld double of a group they can be irrational; in the case of the symmetric group it has been conjectured by Susan Montgomery and her student Rebecca Courter [1] that the higher Frobenius-Schur indicators are always nonnegative (they are known to be integers in this case). It has been shown by Scharf [3] that this is true for representations of the symmetric groups themselves. This result can be interpreted as saying that the class functions counting k-th roots of an element of the symmetric group S_n are always characters. For the double of the symmetric group we know [4] that the indicators are nonnegative for symmetric groups up to S_{25}. However, this is a purely experimental finding

based on heavy computer calculations, and no good *ansatz* for understanding the general statement seems to have surfaced. Several avenues present themselves for trying to understand the situation better: First, in the case of the symmetric group itself the character that counts k-th roots has an explicit description in terms of Lie characters of the symmetric group [5]. The nonnegativity statement for the Drinfeld double can also be read as saying that a (more involved) counting formula gives a character (of the Drinfeld double, described in terms of characters of element centralizers). Is there a description of this character parallel to the description by Lie characters in the group case? Can one at least attempt to find such a description in experimental fashion for small cases? There is already an intriguing relation of the result for the symmetric group to the Drinfeld double: By their construction, the Lie characters are underlying group characters of representations of the double, so that the character formulas that solve the problem for the group itself are structurally related to the double. Understanding this better might help to lift the solution to the double. An entirely different avenue to attack the problem lies in the paper [4] where a novel way to calculate the indicators for Drinfeld doubles is presented. It differs from previous formulas by working more on the level of conjugacy classes of groups than by brutal sums over all group elements. The techniques in [4] made large groups beyond S_{10} accessible for computer calculations in the first place. They also allow to investigate rationality properties of indicators without calculating them (but again with computer help). They have perhaps not been sufficiently exploited for calculations ``by hand'' and questions of positivity.
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As mentioned at the outset, the positivity problem for indicators of the representations of the Drinfeld double of a symmetric group is intriguing (a unsolved quantum group inspired combinatorial/representation theoretic problem on symmetric groups) but there is a risk that it will elude a final solution or even substantial progress. For the prospects of a successful thesis it is reassuring that the problem sits in the vicinity of a host of somewhat related computational and theoretical problems on indicators, more general invariants, or more generally the structure theory of fusion categories. I will allude to some of them below; I am entirely confident that progress on some of these problems should be attainable, and that insight into any of them would benefit our understanding of all of them.

- It has been proved by Guralnick-Montgomery [2] that the Drinfeld doubles of finite Coxeter groups are *totally orthogonal*, that is to say their second indicators are all positive, which means equal to one in this case; this property of the second indicators for group representations is equivalent to the representation being real (i.e. by orthogonal real matrices in a suitable basis). Finite Coxeter groups are also *strongly real*, meaning all their elements are the product of at most two involutions. The strong reality property is useful in proving the total orthogonality of the double for the Coxeter groups but the proof also uses very specific theory of reflection groups. In general, the properties of total orthogonality and strong reality of a finite group are independent. By contrast it has been proved [6] that they are equivalent for simple groups; the proof, however, uses the classification of simple groups and ``inspection''. In work in progress with a PhD student we hopefully show that the double of G=PSL(2,q) is totally orthogonal whenever G itself is. Again, the strong reality is helpful for the proof. Computer experiments (of very limited scope due to the size and complexity of the problem) seem to indicate that the double of a simple group may be totally orthogonal if the group itself is. The techniques to show this are already different between the PSL(2,q) case and the case of reflection groups (reflection group theory vs the linear algebra of two-by-two matrices over finite fields). The common point of strong reality is intriguing, however, all the more because the link between total orthogonality and strong reality does not seem to be conceptually understood for simple groups, but plays an instrumental role in the proofs for cases that we know. It would be very interesting to pursue further examples of this, be it by human or machine calculations.

- Still regarding the indicators for the doubles of finite simple groups, limited computer experiments based on [4] seem to indicate that the higher indicators for PSL-groups are again nonnegative integers. Already for PSL(2,q) we have no good explanation for this even in the totally orthogonal case; the techniques specific to second indicators do not carry over easily. On the other hand, the problem might be more attainable than the analogue for symmetric groups (because it is based in the linear algebra of two-by-two matrices again, albeit being more complicated than for second indicators).
- The representation categories of Drinfeld doubles are special cases of group theoretical categories obtained from a group-subgroup pair. Work in progress (joint with Kayla Orlinsky) shows that in certain situations there are nonnegativity results for second indicators in those categories for finite Coxeter groups (so in a situation similar to that treated by Guralnick-Montgomery but also quite different). Notably no negative second indicators can occur for inclusions of a parabolic subgroup. The result is obtained purely by computer inspection for the exceptional cases however, so there is no conceptual explanation in terms of reflection groups. Also negative indicators do occur for non-parabolic reflection subgroups, and in a class of more general inclusions (including the doubles treated by Guralnick-Montgomery) there seems to be a pattern for the behavior based on combinatorial properties of the Coxeter diagrams involved. This awaits a more detailed analysis by computer experiments, or better yet to be freed from computer help by a more conceptual treatment.
- More general group theoretical categories are obtained from group inclusions equipped with certain group cohomology data. Here conceptual results are more scarce, but at least GAP code has been developed in Dijon to treat related questions computationally.

[1] Rebecca Courter. Computing higher indicators for the double of a symmetric group. ProQuest LLC, Ann Arbor, MI, 2012. Thesis (Ph.D.)–University of Southern California.
[2] Robert Guralnick and Susan Montgomery. Frobenius-Schur indicators for subgroups and the Drinfel′ d double of Weyl groups. Trans. Amer. Math. Soc., 361(7):3611–3632, 2009.
[3] Thomas Scharf. Die Wurzelanzahlfunktion in symmetrischen Gruppen. J. Algebra, 139(2):446–457, 1991.
[4] P. Schauenburg. Higher Frobenius-Schur indicators for Drinfeld doubles of finite groups through characters of centralizers. ArXiv e-prints, April 2016.
[5] Manfred Schocker. On the root symmetry of higher Lie characters. Arch. Math. (Basel), 80(4):337–346, 2003.
[6] C. Ryan Vinroot. Totally orthogonal finite simple groups. Math. Z., 294(3-4):1759–1785, 2020.

Connaissances et compétences requises :

Algèbre : Algèbre linéaire approfondie, notions de la théorie des groupes et représentations, des anneaux et modules.