On dimensions of CAT(0) and CAT(-1) complexes with the same hyperbolic group action. *

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Introduction

An important invariant of a torsion free group G is its geometric dimension, the minimal dimension of a contractible CW-complex on which G acts freely and properly discontinuously. In the case of groups with torsion (where geom. dim. is infinite) it is more convenient to relax the freeness condition and consider finite dimensional actions with finite point stabilizers. Again, the minimal dimension of such a space is of interest. See for example [3]. Currently there is much interest in studying groups which admit actions on spaces of (global) non-positive curvature. The geometrical properties of these spaces offer a great deal of useful information about the group structure. In considering groups which admit such an action it is natural to expect that the minimal dimension of a space of non-positive (or negative) curvature on which G acts (in a reasonable fashion) is greater than the minimal dimension achievable without the same curvature constraint. Our purpose in this paper is to give examples which illustrate this principle. Our examples are very concrete, arising as straightforward quotients of hyperbolic triangle groups.

We consider cocompact group actions on metric polyhedra. A metric polyhedron is understood to have global non-positive or negative curvature if it is CAT(0) or CAT(-1) in the sense of comparison of triangles. More details are given in Section 1. We say that a group action on a metric space is *geometric* if it is isometric, properly discontinuous and cocompact.

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Main Theorem. *1. There exist groups which act properly discontinuously and cocompactly on contractible 2-dimensional polyhedra and geometrically on 3-dimensional CAT(0) polyhedra, but which do not act geometrically on any 2-dimensional CAT(0) polyhedra.*

2. (M. Kapovich, [14]) There exist groups which act geometrically on 2-dimensional CAT(0) polyhedra and on 3-dimensional CAT(-1) polyhedra, but which do not act geometrically on any 2-dimensional CAT(-1) polyhedra.

We give three examples of groups satisfying the statement (1), each with slightly different properties. Using the notation explained in Section 2, these are:

$G(5,3,4)$: We give a 3-dimensional CAT(0) polyhedron on which this group acts geometrically and which contains isometrically embedded copies of the flat plane $\mathbb{E}^2$ (Proposition 3.3). Hence the group is not δ-hyperbolic (by [6], see also Theorem 3.1, p.459, [8]).

$G(5,3,5)$: In this case, the group acts geometrically on a 3-dimensional CAT(-1) polyhedron and hence is δ-hyperbolic.

$G(3,5,3)$: We give a 3-dimensional CAT(0) polyhedron on which this group acts geometrically but which does not obviously admit any CAT(-1) metric, and yet does not contain any isometrically embedded $\mathbb{E}^2$ (Proposition 3.3). By the Theorem of [6] the group is δ-hyperbolic. The following is motivated by a question of Gromov as to whether every δ-hyperbolic group acts cocompactly by isometries on a CAT(-1) space.

Question. Does the δ-hyperbolic group $G(3,5,3)$ admit any properly discontinuous cocompact action by isometries on a finite dimensional polyhedron of negative curvature? If so, in what dimension?

We give four examples of groups satisfying statement (2) of the Main Theorem. In the notation of Section 2, these are the groups $G(6,3,3)$, $G(6,3,4)$, $G(6,3,5)$, and $G(4,4,3)$. The first of these was the example considered by M. Kapovich [14].

Since this work was begun several papers [6, 4, 5, 9, 10] have treated the question of CAT(0) versus geometric dimension, giving examples of torsion free groups satisfying statement (1) of the Main Theorem, or similar. The techniques involved in those papers are quite different to those used here where the existence of torsion elements play an important role. The results of [6, 4, 9, 10] concern non-hyperbolic CAT(0) groups and rely systematically on the existence of $\mathbb{Z}^2$ subgroups. On the other hand, the hyperbolic group examples presented in [5] are based on a study of a torsion free subgroup of the Kapovich example $G(6,3,3)$, for which statement (2) of the Main Theorem holds. By taking HNN extensions of this group we exhibited, in that paper, an infinite family of 2-dimensional hyperbolic groups with no geometric action on any 2-dimensional CAT(0) space.

All the examples presented in this paper are full symmetry groups of certain highly regular simply-connected 2-dimensional polyhedra studied by Haglund in [13] and by Ballman and Brin in [1]. These polyhedra ($\widetilde{Z} = \widetilde{Z}(r,m,p)$ in our notation, with symmetry group $G(r,m,p)$) are in each case composed of infinitely many copies of a regular r-gon (triangles, squares, pentagons or hexagons in our examples) glued along their edges in such a way that the link of every vertex is a graph isomorphic to the 1-skeleton of a platonic solid (the tetrahedron for $G(r,3,3)$, octahedron for $G(r,3,4)$, icosahedron for $G(r,3,5)$, cube for $G(r,4,3)$, and dodecahedron for $G(r,5,3)$). According to [1], Theorem 3, this much information determines, up to isomorphism, exactly two polyhedra Π subject also to the following regularity condition: for any pair of vertices $u, v \in \Pi$ and any isomorphism $\phi : \mathrm{Lk}(u) \to \mathrm{Lk}(v)$, there is an automorphism Φ of Π inducing ϕ. The two cases are distinguished by the subgroup of automorphisms which stabilize a single r-gon. This can either be $D_{2r} \times C_2$ or D_{4r}. Our examples are all of the latter 'twisted' kind.

Section 1 presents a brief summary of the theory of complexes of groups [12], [8]. This provides the most convenient language with which to describe and study the groups and complexes which interest us in this paper.

In Section 2 we place the examples already discussed in the context from which we feel they arise most naturally. We view them as belonging to the class of groups $G(r,m,p)$ which are quotients of hyperbolic triangle groups (associated with regular tilings of $\mathbb{H}^2$) by a single element which is a power of a glide reflection. These groups are of some interest in their own right, and various further observations concerning them are collected in the Appendix.

In Sections 3 and 4 we consider only the groups $G(r,m,p)$ for which $\frac{1}{m}+\frac{1}{p}>\frac{1}{2}$. Thus $(m,p) = (3,3)$, $(3,4)$, $(3,5)$, $(4,3)$, or $(5,3)$.

We investigate, in Section 3, a family of 3-dimensional polyhedra $\widetilde{Y}$ on which these groups act properly discontinuously. Each of these polyhedra is built from many copies of a 3-dimensional polytope $\Pi(m,p)$ associated with the platonic solid with face valence m and vertex valence p. The Haglund-Ballman-Brin polyhedra $\widetilde{Z}$, described above, then arise naturally as 2-spines of these spaces. We determine which of these 3- and 2-dimensional polyhedra admit G-invariant CAT(0) or CAT(-1) polyhedral metrics, and observe an 'improvement' in curvature resulting from a 'fattening' of the complexes from dimension 2 to dimension 3.

Finally, in Section 4 we extend the original argument of Kapovich [14] in order to show that, in terms of finding polyhedra of non-positive curvature on which these particular groups act, the polyhedra $\widetilde{Z}$ are the best one can hope for in dimension 2. This, together with the comparisons made in Section 3, leads to the statement of the Main Theorem.

1 Complexes of groups and non-positive curvature

We outline, briefly, the elements of the theory of complexes of groups which we shall need. A fuller treatment can be found in [12] or [8]. This theory is a direct generalisation of the Bass-Serre theory of graphs of groups and group actions on trees.

For our purposes, let X be a finite connected simplicial complex (the theory of [12] applies more generally to (possibly infinite) simplicial cell complexes).

A *complex of groups* $G(X)$ *on* X consists of a group G_σ for each simplex σ in X together with an injective homomorphism $\phi_{\tau\sigma} : G_\sigma \to G_\tau$ for every pair of simplexes $\tau \subset \sigma$ in X, satisfying the following conditions. Wherever $\mu \subset \tau \subset \sigma$ in X the injections $\phi_{\mu\sigma}$ and $\phi_{\mu\tau}\phi_{\tau\sigma}$ differ by an inner automorphism, or *holonomy*, of G_μ, and these holonomies satisfy a certain cocycle condition which arises in dimension 3 and above. In our examples, we shall take all holonomies to be trivial (that is $\phi_{\mu\sigma} = \phi_{\mu\tau}\phi_{\tau\sigma}$ always) in which case the cocycle condition is automatically satisfied, so need not concern us in this paper.

Given a connected complex of groups $G(X)$ one may define combinatorially its *fundamental group*, $\pi_1(G(X))$. Rather than give the details of the general definition of this group (which may be found in [12]), we simply observe that in the cases we shall consider, where X is a simply-connected complex, $\pi_1(G(X))$ is given by:

Generators: the elements of G_σ for $\sigma \in X$, and

Relations: those of the G_σ together with the relations $x = \phi_{\tau\sigma}(x)$ for all $x \in G_\sigma$ and for every $\tau \subset \sigma$ in X.

If a group G acts without inversion (which is to say that any simplex which is stabilised by $g \in G$ is fixed pointwise by g) on a simplicial complex $\overline{X}$, then one can associate to this action a complex of groups $G(X)$ over the quotient $X = G\backslash\overline{X}$, which is unique up to isomorphism (see [12] for details). For σ in X the group G_σ is isomorphic to the stabilizer in G of any lift $\overline{\sigma}$ of σ in $\overline{X}$. Complexes of groups which arise in this way are said to be *developable*. One has (see [12]):

Theorem 1.1. *A complex of groups $G(X)$ is developable if and only if the natural map $G_\sigma \to \pi_1(G(X))$ is injective for every $\sigma \in X$.*

Moreover, if $G(X)$ is developable then it is canonically associated to an action of $\pi_1(G(X))$ without inversion on a simply connected simplicial complex $\widetilde{X}$. The complex $\widetilde{X}$ is referred to as the universal covering complex of $G(X)$.

The important point is that, for $G(X)$ a developable complex of groups, the structure of $\widetilde{X}$ is completely determined by local information. The so-called *orbihedron* structure on an arbitrary complex of groups $G(X)$ associates to each σ in X the simplicial link $\mathrm{Lk}\widetilde{\sigma}$ which one expects to see about any lift $\widetilde{\sigma}$ of σ in the universal cover (if it were to exist). In the cases we shall consider, where X is simply connected and all holonomies are trivial, $\mathrm{Lk}\widetilde{\sigma}$ will be just the derived complex of the partially ordered set $\mathcal{L}(\sigma)$, where $\mathcal{L}(\sigma)$ denotes the set of all cosets in G_σ of the subgroups $\phi_{\sigma\xi}(G_\xi)$ for all simplexes ξ containing σ, and which is ordered by inclusion.

By an *M_χ-polyhedron*, for fixed $\chi \in \mathbb{R}$, we mean the underlying polyhedron of a simplicial complex which has been endowed with a piecewise metric in which every simplex is isometric to a geodesic simplex in M_χ, the complete simply connected Riemannian manifold with constant curvature χ. An M_χ-polyhedron is a CAT(χ) polyhedron if it satisfies the CAT(χ)-inequality globally, namely:

Any geodesic triangle (if $\chi \leqslant 0$), or any geodesic triangle of perimeter less than $\frac{2\pi}{\sqrt{\chi}}$ (if $\chi > 0$), is "no fatter" than a comparison triangle of the same sidelengths in M_χ. "No fatter" means that points on a triangle are no further apart than the corresponding points on the comparison triangle.

Suppose now that $G(X)$ is a graph of groups over an M_χ-polyhedron X. Then the links in the orbihedron structure on X are naturally M_1-polyhedra with the angular metric induced from the metric on X. The following Theorem is of key importance:

Theorem 1.2 ([12], p.528). *Suppose that $G(X)$ is a graph of groups over an M_χ-polyhedron X, with $\chi \leqslant 0$, and that $\mathrm{Lk}\widetilde{\sigma}$ is a CAT(1) polyhedron for every vertex σ in X. Then $G(X)$ is developable and, moreover, with the M_χ-polyhedral metric induced from X, the universal covering complex $\widetilde{X}$ has the structure of a CAT(χ) polyhedron, on which the fundamental group acts by isometries.*

If it satisfies the hypothesis of the Theorem, $G(X)$ is said to be *non-positively curved*, and *negatively curved* if moreover $\chi < 0$.

In testing whether a complex of group is non-positively (or negatively) curved, we use the fact that an M_1-polyhedron L is CAT(1) if and only if (i) the link of every vertex of L is also CAT(1); and (ii) every closed geodesic in L has length $\geqslant 2\pi$. When L is 1-dimensional (a metric graph), this means precisely that every simple closed curve has length $\geqslant 2\pi$ (see [8]).

Using the fact that $\widetilde{X}$ is CAT(0) one deduces, as a corollary to Theorem 1.2, the following useful result (see [12], p.529):

Proposition 1.3. *If $G(X)$ is a non-positively curved complex of groups then any finite subgroup of $\pi_1(G(X))$ is conjugate to a subgroup of some G_σ.*

Finally, we make the following observation that is relevant to our purposes:

Proposition 1.4. *If $G(X)$ is a compact complex of groups with finite vertex groups G_σ and which is developable, then $\pi_1(G(X))$ acts properly discontinuously and cocompactly on $\widetilde{X}$. In particular when $G(X)$ is non-positively curved then $\pi_1(G(X))$ acts geometrically on the CAT(χ) polyhedron $\widetilde{X}$, where $\chi \leqslant 0$.*

2 Quotients of planar tiling symmetry groups by glides

Let $T(r,m)$ denote a tiling of $\mathbb{H}^2$ by regular m-gons with angle π/r (so that the valence of each vertex is $2r$). Such a tiling exists if and only if $\frac{1}{m} + \frac{1}{2r} < \frac{1}{2}$. The full symmetry group of $T(r,m)$ is just the triangle group $\Delta(2r,m,2)$. Let γ denote a glide reflection whose axis is one of the geodesics in the 1-skeleton of the tiling $T(r,m)$, and whose translation distance is equal to the length of a side of a tile in $T(r,m)$. This element is well-defined up to conjugacy. For positive integers $p \geqslant 3$ (and m, r satisfying $\frac{1}{m} + \frac{1}{2r} < \frac{1}{2}$) we define the groups

$$G(r,m,p) \;=\; \Delta(2r,m,2)/\langle\langle \gamma^p \rangle\rangle\,.$$

Here $\langle\langle \gamma^p \rangle\rangle$ denotes the smallest normal subgroup of $\Delta(2r,m,2)$ which contains γ^p. Thus $G(r,m,p)$ is just the full symmetry group of the orbit space

$$S(r,m,p) = T(r,m)/\langle\langle \gamma_p \rangle\rangle\,.$$

Note that $S(r,m,p)$ need not in general be a surface, nor should one expect that any complete tile of $T(r,m)$ is preserved under the quotient (for example, $G(2,7,3)$ is cyclic of order 2 acting on an isoceles triangle comprising a seventh of the right-angled hyperbolic 7-gon). However, for all but finitely many cases, it will turn out that the space $S(r,m,p)$ is an infinite surface tiled by regular m-gons and that the image of each geodesic in the 1-skeleton of $T(r,m)$ is a simple closed geodesic of edge length p in $S(r,m,p)$.

In order to clarify such a statement, it helps to give the following description of $G(r,m,p)$ as the fundamental group G of a complex of groups $G(X)$ over the complex X which consists of a pair of triangles PQR and MQR as illustrated in Figure 1. We assign a group G_σ to each simplex σ in X as follows. (For simplicity, we view the injections $G_\sigma \to G_\tau$ for $\tau \subset \sigma$ in X as group inclusions. This should cause no confusion, as no ambiguities will arise in this example).

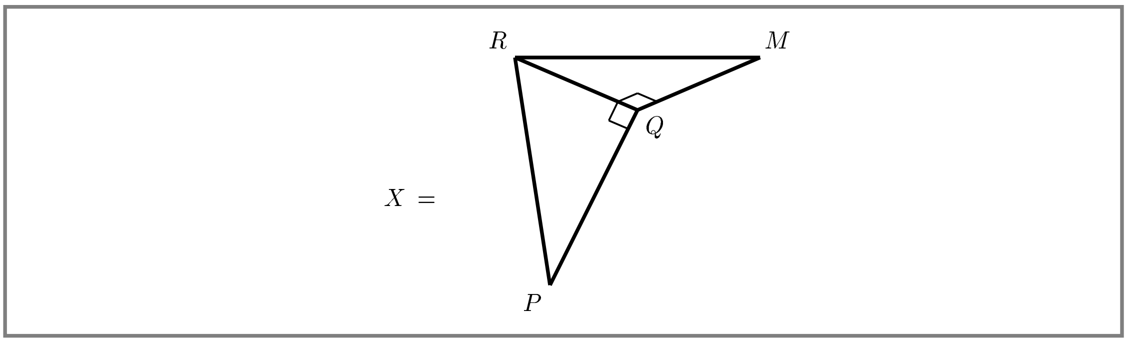


Figure 1: The complex of groups $G(X)$.

$$\begin{aligned} G_{MQR} &= \langle 1 \rangle\,,\ G_{QR} = \langle a | a^2 \rangle\,,\ G_{RM} = \langle b | b^2 \rangle\,,\ G_{MQ} = \langle c | c^2 \rangle\,,\\ G_M &= \langle b, c | b^2, c^2, (bc)^m \rangle\,,\ G_Q = \langle a, c | a^2, c^2, (ac)^2 \rangle\,,\ G_R = \langle a, b | a^2, b^2, (ab)^{2r} \rangle\,,\\ G_{PQR} &= G_{QR} = C_2\langle a \rangle\,,\ G_{PQ} = G_Q = C_2\langle a \rangle \times C_2\langle c \rangle\,,\\ G_{PR} &= C_2\langle a \rangle \times C_2\langle \rho \rangle\,,\ \text{where } \rho = (ab)^r\\ G_P &= C_2\langle a \rangle \times \langle \rho, c | \rho^2, c^2, (\rho c)^p \rangle\,. \end{aligned}$$

The fundamental group of this complex of groups has presentation

$$\pi_1(G(X)) = \langle a, b, c | a^2, b^2, c^2, (ab)^{2r}, (bc)^m, (ac)^2, ((ab)^r c)^p \rangle\,,$$

which is clearly a presentation for the quotient $G(r,m,p)$ of the triangle group $\Delta(2r,m,2)$ where $(ab)^r c = \rho c$ represents the glide γ along an edge of the $T(r,m)$ tiling.

We suppose for the moment that the complex of groups $G(X)$ is developable, in which case the universal covering complex $\widetilde{X}$ (which is unique, by Theorem 1.1) has the following description. The union of all translates of the triangle MQR forms a subcomplex $S \subset \widetilde{X}$ which is isomorphic to $S(r,m,p)$, and which, from the local information, is necessarily a surface tiled by translates of the m-gon $G_M(MQR)$. This tiling corresponds precisely to the image in $S(r,m,p)$ of the tiling $T(r,m)$. The whole complex $\widetilde{X}$ consists of S together with translates of the p-gon $G_P(PQR)$, one of which is attached along each closed path in S which corresponds to the image in $S(r,m,p)$ of a geodesic line in the 1-skeleton of the tiling $T(r,m)$ of $\mathbb{H}^2$. These attaching paths are in fact closed circuits of edge length p in the 1-skeleton of the polygonal tiling on S. Note also that, because γ is an orientation reversing isometry, the surface $S(r,m,p)$ is orientable if and only if p is even, and that when p is odd the attaching paths for the p-gons are orientation reversing paths in $S(r,m,p)$.

Note that $G(r,m,p)$ is finite if and only if $S(r,m,p)$ is finite. If $G(X)$ happens to be non-positively curved, then $G(r,m,p)$ is infinite (since it is not conjugate to any G_σ, for $\sigma \in X$ — see Proposition 1.3) and hence both $\widetilde{X}$ and the surface $S(r,m,p)$ are infinite complexes.

We turn now to our investigation of the developability (or rather, the "curvature") of the complex of groups $G(X)$ just introduced. Let PQR and MQR be Euclidean triangles each with a right angle at the vertex Q and with angles $\frac{\pi}{m}$ and $\frac{\pi}{p}$ at the vertices M and P respectively. With this metric all links $Lk\widetilde{\sigma}$ in the orbihedron associated to $G(X)$ are obviously CAT(1), with the possible exception of $Lk\widetilde{R}$ which is as shown in Figure 2, with $\theta = \angle MRQ = (\frac{1}{2} - \frac{1}{m})\pi$ and $\phi = \angle PRQ = (\frac{1}{2} - \frac{1}{p})\pi$.

We defer the proof of the following propositions to the Appendix.

Proposition 2.1. *Let $G = G(r,m,p)$ with $p \geqslant 3$ and $\frac{1}{m} + \frac{1}{2r} < \frac{1}{2}$. With the metric just described, the complex of groups $G(X)$ with fundamental group G is non-positively curved (so that G acts geometrically on the 2-dimensional CAT(0) polyhedron $\widetilde{X}$) in all except the following cases:*

1. *$1/m + 1/p > 1/2$ (so $(m,p) = (3,3), (3,4), (3,5), (4,3),$ or $(5,3)$);*
2. *$r = 2$ and (m,p) is one of the following 16 exceptional pairs:*

   $(5,p)$ *for* $3 \leqslant p \leqslant 9$,

   $(6,3), (6,4), (6,5)$,

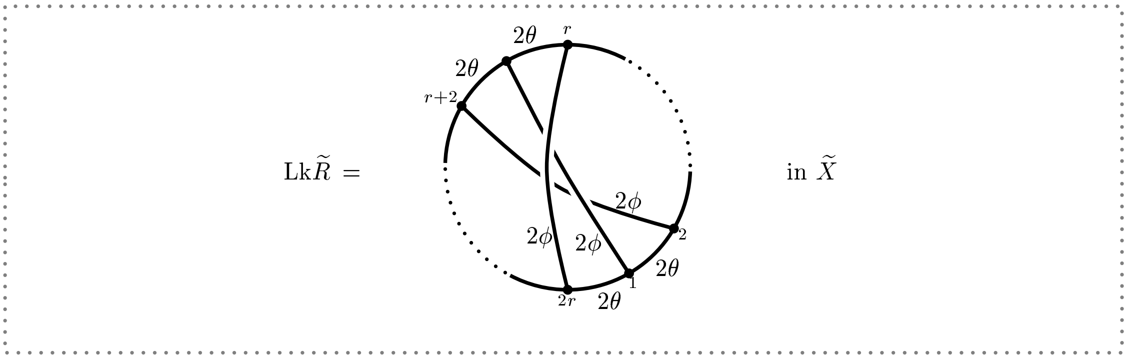


Figure 2: The link complex over the vertex R in $G(X)$.

$(7,3), (7,4)$ *and*

$(m,3)$ *for* $8 \leqslant m \leqslant 11$.

Note that $G(2,5,3)$ *alone is covered by both (1) and (2).*

If there is sufficient freedom that Lk$\widetilde{R}$ is still CAT(1) for some values of θ and ϕ strictly less than $(\frac{1}{2}-\frac{1}{m})\pi$ and $(\frac{1}{2}-\frac{1}{p})\pi$ respectively, then the metric on $G(X)$ may be deformed so that PQR and MQR are both triangles of constant negative curvature ($\chi = -1$), with angles $(\frac{\pi}{m}, \frac{\pi}{2}, \theta)$ and $(\frac{\pi}{p}, \frac{\pi}{2}, \phi)$, respectively. In this case $\widetilde{X}$ is a CAT(-1) polyhedra on which $G(r,m,p)$ acts properly discontinuously and cocompactly by isometries.

Proposition 2.2. *The CAT(0) 2-complex* $\widetilde{X}$ *(of Proposition 2.1) contains isometrically embedded flat planes* $\mathbb{E}^2$*, implying that* $G(r,m,p)$ *is not word hyperbolic, in each of the following cases:*

$$G(r,3,6) \text{ with } r \geqslant 4$$
$$G(r,4,4) \text{ and } G(r,6,3) \text{ with } r \geqslant 3$$
$$G(2,8,4) \text{ and } G(2,6,6)\,.$$

In all other cases where $G(X)$ *is non-positively curved (see Proposition 2.1) the metric may be deformed so that* $G(X)$ *has strictly negative curvature, and* $\widetilde{X}$ *is a CAT(-1) polyhedron. In particular* $G(r,m,p)$ *is word hyperbolic.*

In the Appendix we also show that when $G(X)$ is non-positively curved then $G(r,m,p)$ is 1-ended. *Some of the 16 exceptional cases of Proposition 2.1(2) are also discussed there.*

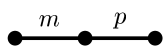
This leaves us to consider, in the following sections, the groups which arise in case (1) of Proposition 2.1, namely the case where $1/m + 1/p > 1/2$. We note that if $1/m + 1/p \leqslant 1/2$ ($r \geqslant 3$ say) we can identify certain planar subcomplexes of $\widetilde{X}$, which happen to be flat (so $\cong \mathbb{E}^2$) precisely when $1/m + 1/p = 1/2$. These planes have a 'chequerboard' tiling by m-gons and p-gons, and correspond to the subgroup of $G(r,m,p)$ generated by the elements b, c, and ρ (note that the relation $(b\rho)^2 = 1$ holds in the vertex group G_R). When $1/m + 1/p > 1/2$, the corresponding object is a tiled 2-sphere which appears (generically) as a subcomplex $\widetilde{X}$. We now study the polyhedra that are obtained by filling each such sphere in $\widetilde{X}$ with a 3-dimensional polytope.

3 The nonpositively curved 3-complexes

Throughout this section we suppose that $\frac{1}{m} + \frac{1}{p} > \frac{1}{2}$. Let $P(m,p)$ denote the platonic solid with face valence m and vertex valence p — i.e: the tetrahedron $P(3,3)$, cube $P(4,3)$, octahedron $P(3,4)$, dodecahedron $P(5,3)$, or icosahedron $P(3,5)$.

We define the Euclidean polytope $\Pi = \Pi(m,p)$ which is obtained from $P(m,p)$ via the median construction. Namely, one takes the convex hull of the set of midpoints of edges on the boundary of $P(m,p)$. The result is a polytope whose boundary is tiled alternately by m-gons (from the faces of $P(m,p)$) and p-gons (from the lopped-off vertices of $P(m,p)$), otherwise known as the octahedron ($\Pi(3,3)$), cuboctahedron ($\Pi(3,4)$ and $\Pi(4,3)$), or icosadodecahedron ($\Pi(3,5)$ and $\Pi(5,3)$). We remember, however, that the polytope Π, via its construction, comes equipped with a colouring of its boundary faces: "red" for the m-gons (corresponding to faces of $P(m,p)$) and "green" for the p-gons (corresponding to vertices of $P(m,p)$).

The Coxeter group $W = W(m,p,2)$ having Coxeter graph



acts on Π with fundamental 'chamber' the simplicial polytope Y illustrated in Figure 3. The action is generated by reflections in three 'walls' — the triangle PRC, the flat quadrilateral $PQMC$ (composed of PQC and MQC), and the triangle RMC — and fixes the vertex C which lies at the centre of the polytope Π. Simplicially, Y is a compact cone over X at the point C.

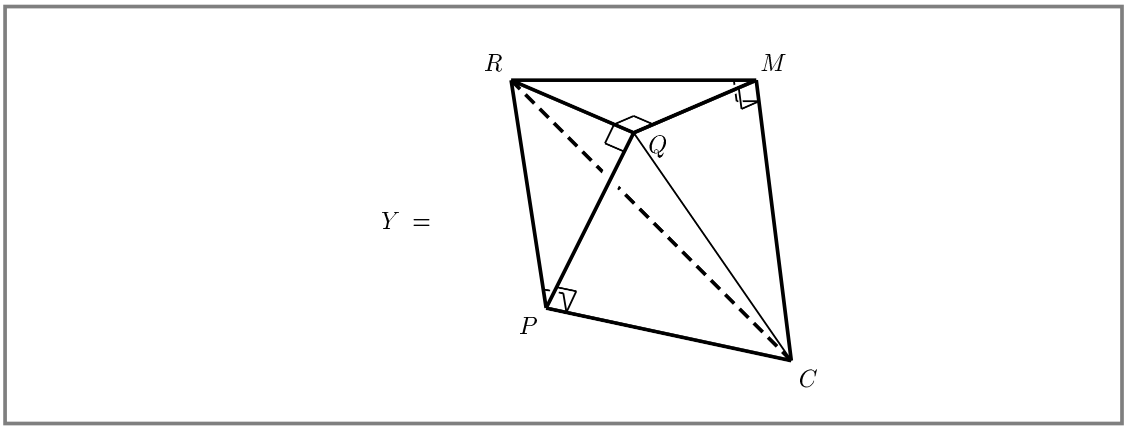


Figure 3: The complex of groups $G(Y)$.

For given (r,m,p) we define the complex of groups $G(Y)$ over Y which restricts, over the subcomplex X, to $G(X)$ (defined in Section 2) and with additional groups as follows:

$$G_C = W = \langle b, c, \rho | b^2, c^2, \rho^2, (bc)^m, (c\rho)^p, (b\rho)^2 \rangle$$
$$G_{MC} = D_{2m}\langle b, c \rangle\ ,\ G_{RC} = C_2\langle b \rangle \times C_2\langle \rho \rangle\ ,\ G_{PC} = D_{2p}\langle c, \rho \rangle$$
$$G_{RMC} = C_2\langle b \rangle\ ,\ G_{QC} = G_{PQC} = G_{MQC} = C_2\langle c \rangle\ ,\ G_{PRC} = C_2\langle \rho \rangle\ ,$$
$$G_{QRC} = G_{PQRC} = G_{MQRC} = \langle 1 \rangle\ .$$

As before we have that $\pi_1(G(Y))$ is just $G(r,m,p)$.

Suppose that $G(Y)$ is developable. Then $G(X)$ is also developable (since $G(X) \subset G(Y)$ with the same fundamental group, see Theorem 1.1). We identify $\widetilde{X}$ with the 2-subcomplex of $\widetilde{Y}$ lying above $X \subset Y$. Around each translate of the vertex C in $\widetilde{Y}$ one sees a copy of Π whose boundary lies in $\widetilde{X}$, the "red" tiles being the translates of the m-gon $G_M(MQR)$ which tile the surface $S \cong S(r,m,p)$ in $\widetilde{X}$, and the "green" tiles being the translates of the p-gon $G_P(PQR)$. One may think of $\widetilde{Y}$ as being obtained from $\widetilde{X}$ by coning off certain combinatorial spheres, one for each translate of the polytope Π in $\widetilde{Y}$. Alternatively $\widetilde{Y}$ may be described as a union of copies of Π with their "green" faces identified in pairs and with r copies of Π adjacent to each shared vertex. In order that the union of "red" tiles form a surface the link of each of these shared vertices must topologically be a Möbius band.

Having just described how the universal cover $\widetilde{Y}$ will look we continue now by determining conditions under which we will know that $G(Y)$ is developable.

Let Y be endowed with the metric induced from the Euclidean polytope Π. Then all vertex links in the orbihedron structure on $G(Y)$ are CAT(1) with the possible exception of $\mathrm{Lk}\widetilde{R}$. ($\mathrm{Lk}\widetilde{C}$ is isometric to S^2, as is $\mathrm{Lk}\widetilde{P}$ being the union of a hemisphere in Π and its image under a; $\mathrm{Lk}\widetilde{M}$ is a hemisphere in Π; and $\mathrm{Lk}\widetilde{Q}$ the orthogonal join of a 0-sphere with a closed interval of length $2\angle PQM$. A careful inspection of the poset $\mathcal{L}(R)$ of cosets in G_R yields the following description for $L = \mathrm{Lk}\widetilde{R}$:

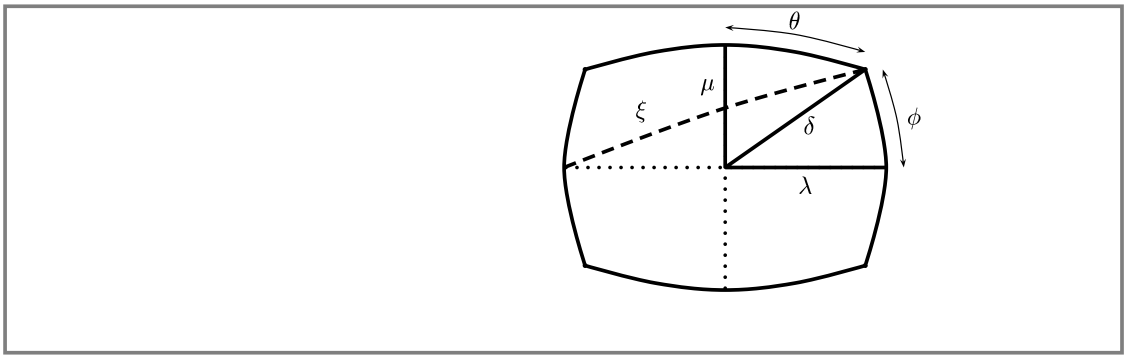


Figure 4: The model spherical quadrilateral Θ.

The link L is composed of equiangular spherical quadrilaterals, each with sidelengths 2θ and 2ϕ (where $\theta = \angle MRQ = (\frac{1}{2} - \frac{1}{m})\pi$ and $\phi = \angle PRQ = (\frac{1}{2} - \frac{1}{p})\pi$, as in Section 2). One of these, which we denote Θ is illustrated in Figure 4. It is precisely isometric to the link of R in Π and is obtained, combinatorially, by considering the action of $G_R \cap G_C = \langle b, \rho \rangle$ on the cosets of $G_{RQ} = \langle a \rangle$, $G_{RM} = \langle b \rangle$, $G_{RP} = \langle a, \rho \rangle$, and $G_{RC} = \langle b, \rho \rangle$ inside the group $G_R = D_{4r}\langle a, b \rangle$. The length δ shown in Figure 4 may be calculated by observing that this angle is the vertex angle of a regular Euclidean n-gon geodesically embedded in Π with boundary lying in the 1-skeleton. Here $n = 4$ for the octahedron $\Pi(3,3)$, $n = 6$ for the cuboctahedron ($\Pi(3,4)$ and $\Pi(4,3)$), and $n = 10$ for the icosadodecahedron ($\Pi(3,5)$ and $\Pi(5,3)$). The remaining dimensions λ, μ and ξ which are indicated in Figure 4 may be computed by using the cosine rule for right angled spherical triangles. Thus $\cos\lambda = \frac{\cos\delta}{\cos\phi}$, $cos\mu = \frac{\cos\delta}{\cos\theta}$ and $\cos\xi = \cos 2\lambda \cos\phi$. The results of these computations are listed in Table 1. The values for λ, μ and ξ which are given in degrees are lower estimates to the nearest degree.

The whole link L, as illustrated in Figure 5, is the union of the translates of Θ by the elements $(ab)^k$ for $k = 1, .., r$. These r copies of Θ are glued end to end along the sides of length 2ϕ to

(m,p)	θ	ϕ	δ	λ	μ	ξ
$(3,3)$	$\frac{\pi}{6}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$> 35°$	$> 35°$	$> 72°(< 75°)$
$(3,4)$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$> 54°$	$\frac{\pi}{2}$
$(4,3)$	$\frac{\pi}{4}$	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$> 54°$	$\frac{\pi}{4}$	$> 105°$
$(3,5)$	$\frac{\pi}{6}$	$\frac{3\pi}{10}$	$\frac{2\pi}{5}$	$> 58°$	$> 69°$	$> 104°$
$(5,3)$	$\frac{3\pi}{10}$	$\frac{\pi}{6}$	$\frac{2\pi}{5}$	$> 69°$	$> 58°$	$> 130°$

Table 1: Dimensions of Θ for different values of (m,p).

form a band. Since $\rho = (ab)^r$ acts on Θ by a reflection (in the midline of the band) the link is a Möbius band, as previously suggested, and as indicated in the figure. The points which lie at the corners of the Θ_j's shall be called *corner points* and are denoted $c(1), c(2), .., c(2r) = c(0)$ in sequence around the boundary ∂L of L, and so that $c(i+r)$ lies directly opposite $c(i)$ (indices taken mod $2r$).

We define the closed geodesic γ_0 of length $2r\lambda$ which follows the midline of L with $\gamma_0(2i\lambda)$ lying on the midpoint between $c(i)$ and $c(i+r)$ for each $i = 1, 2, .., r$. We define γ_1 to be the closed geodesic of length $4r\theta$ which follows ∂L with $\gamma_1(2i\theta) = c(i)$ for $i = 1, 2, .., 2r$.

We call a locally geodesic segment $\sigma : [0,t] \to L$ a *proper segment* if $\sigma(s) \notin \partial L$ for all $0 < s < t$. Note that the universal cover of $\text{int}(L) = L \setminus \partial L$ is embedded locally isometrically as a subspace of the universal cover $\widetilde{S}$ of $S = \mathbb{S}^2 \setminus \{N, S\}$, the 2-sphere with north and south poles removed. Geodesics in the $\widetilde{S}$ are either 'sinusoidal', projecting to great circles in S, or longitudinal. Thus a proper segment in L is either orthogonal to the midline, or lifts to a sinusoidal segment in $\widetilde{S}$.

Let k be the unique positive integer such that $\frac{\pi}{2} - \lambda \leqslant k\lambda < \frac{\pi}{2}$. There is a unique proper segment ν in L from $c(0)$ to $c(k+r)$ with midpoint $m = \gamma_0(k\lambda)$. (Note that the first half of ν is the unique geodesic segment joining $c(0)$ to m inside the spherical right isosceles triangle with base $c(0)\gamma_0(0)$ and apex $\gamma_0(\frac{\pi}{2})$. The rest is obtained by a π-rotation about m. Also $\ell(\nu) < \pi$.) We define the closed geodesic γ_2 in L to be that which follows ν from $c(0)$ to $c(k+r)$ and then follows γ_1 (in the positive direction) from $c(k+r)$ around to $c(0)$ again.

The closed geodesics γ_0, γ_1 and γ_2 are illustrated in Figure 5. Clearly L is locally CAT(1). We now prove the following:

Lemma 3.1. *Let $L = L(r, \theta, \phi)$ denote the piecewise spherical complex which is topologically a Möbius band constructed by gluing r isometric copies of the equiangular quadrilateral Θ (illustrated in Figure 4) along their sides of length 2ϕ. Then L is CAT(1) if and only if each of the closed geodesics γ_0, γ_1 or γ_2 has length $\geqslant 2\pi$.*

Remark. Note that γ_0 which follows the midline of L has length $2r\lambda$ while γ_1 follows the boundary of L and has length $4r\theta$. In the cases we will encounter, the remaining curve is

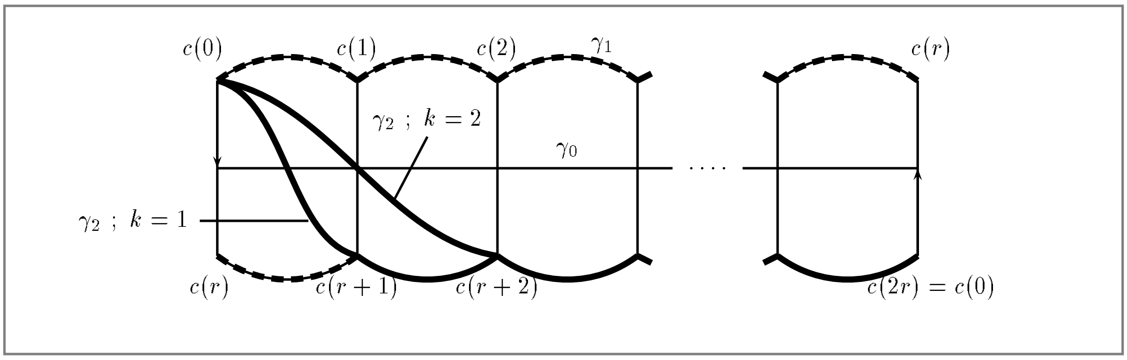


Figure 5: Candidates for the systole in $\mathrm{Lk}\widetilde{R}$.

measured as follows: $\ell(\gamma_2) = 2(r-1)\theta + 2\delta$ if $45 \leqslant \lambda < 90$ ($k = 1$), and $\ell(\gamma_2) = 2(r-2)\theta + 2\xi$ if $30 \leqslant \lambda < 45$ ($k = 2$).

Proof. The complex L is a compact, locally CAT(1) geodesic space, and so fails to be CAT(1) if and only if there exists a closed geodesic of length $< 2\pi$ (see, for example, Proposition II.4.16 of [8]). We may assume moreover that such a geodesic has length equal to the systole, sys(L) =(infimum of lengths of closed geodesics in L). The necessity of the condition that $\ell(\gamma_i) \geqslant 2\pi$ for $i = 0, 1, 2$ is now clear. We now show sufficiency.

Suppose that γ is a closed geodesic in L such that $\ell(\gamma) = \mathrm{sys}(L) < 2\pi$. We either show that γ is at least as long as one of the curves γ_0, γ_1 or γ_2, or find a homotopically nontrivial rectifiable loop of length strictly less than $\ell(\gamma)$, giving a contradiction. The contradiction follows from the fact that, since L is compact, every homotopically nontrivial rectifiable loop σ 'shrinks' to a closed geodesic of length at most $\ell(\sigma)$ (see [2], Ch.3).

Consider first what happens if γ is disjoint from γ_0. Projecting orthogonally away from γ_0 onto ∂L defines a continuous map $f : L \setminus \gamma_0 \to \partial L$ which is locally distance reducing. Thus γ is mapped to a closed curve in the boundary whose length is at most the length of γ. But the shortest closed path in the boundary is γ_1.

If, on the other hand, γ intersects γ_0 we may suppose that all intersections are transverse, for otherwise γ is constrained to follow γ_0 and so has length a multiple of $\ell(\gamma_0)$ (geodesic segments are uniquely extendible in int(L)). If γ crosses γ_0 transversely at distinct points p and q, then reflecting a segment between p and q in the midline gives a loop of the same length with kinks at p and q and which may be locally shortened at these points, giving a rectifiable loop σ of length strictly less than $\ell(\gamma)$. Thus we may assume that γ intersects γ_0 transversely at a single point p.

If γ is disjoint from ∂L, then we must have $\ell(\gamma) = \ell(\gamma_0) = \pi$ with γ and γ_0 cobounding a spherical sector. (This is best seen by lifting to the universal cover of L).

We may suppose now that γ intersects ∂L. Note that the only points at which a closed geodesic may enter or leave ∂L are at the corner points. Thus γ contains at least a proper segment α with endpoints on corner points and which passes through p. By projecting the rest of γ to the boundary we may assume that α is the only part of γ which intersects int(L). Without loss, we suppose that α starts at $c(0)$.

If $\ell(\alpha) > \pi$ then a subsegment of length π may be 'rotated', fixing its endpoints, to give a new path of the same length with at least one kink which admits shortening to a rectifiable loop of length strictly less than $\ell(\gamma)$. If $\ell(\alpha) < \pi$ then α can only be ν and γ is precisely γ_2. The only remaining possibility is that α is a proper segment of length π from $c(0)$ to $c(k+1+r)$, in the case that $(k+1)\lambda = \frac{\pi}{2}$. But in this case it is easily seen that γ and γ_2 have the same length. $\square$

Proposition 3.2. *With the given Euclidean metric, $G(Y)$ is nonpositively curved, and $G(r,m,p)$ acts properly discontinuously and cocompactly by isometries on the CAT(0) polyhedron $\widetilde{Y}$, in the following cases:*

$$\begin{array}{ll} G(r,3,3) & \textit{with } r \geqslant 6 \\ G(r,3,4) & \textit{with } r \geqslant 5 \\ G(r,4,3) & \textit{with } r \geqslant 4 \\ G(r,3,5) & \textit{with } r \geqslant 5 \\ G(r,5,3) & \textit{with } r \geqslant 3\,. \end{array}$$

Moreover, in all but the two cases $G(5,3,4)$ and $G(3,5,3)$ the metric on Y may be modified so that $G(Y)$ is negatively curved and $\widetilde{Y}$ is a CAT(-1) polyhedron.

Proof. In general, $G(Y)$ is non-positively curved (with the given M_0-polyhedral metric) if and only if the link $L = \mathrm{Lk}\widetilde{R}$ is CAT(1). Note firstly that in all the cases listed the midline γ_0 in L has length $2r\lambda$ (where the values of λ may be read from Table 1) which is well in excess of 2π (in fact, more than 400° in all cases). Thus, by Lemma 3.1, $G(Y)$ is non-positively curved if and only if the closed curves γ_1 and γ_2 of Figure 5 are both at least 2π in length. Since $\theta \geqslant \frac{\pi}{6}$ in all cases (see Table 1), we have $\ell(\gamma_1) = 4r\theta \geqslant 2\pi$ whenever $r \geqslant 3$. It remains to investigate the length of γ_2.

In the case $(m,p) = (3,3)$ we have $k = 2$ $(30° \leqslant \lambda < 45°)$ and so $\ell(\gamma_2) = 2(r-2)\theta + 2\xi$. From Table 1 we have $\theta = 30°$ and $72° < \xi < 75°$. Thus, $\ell(\gamma_2)$ is strictly greater than 2π for all $r \geqslant 6$ and strictly less than 2π otherwise.

In the remaining four cases, $k = 1$ $(45° \leqslant \lambda < 90°)$ and $\ell(\gamma_2) = 2(r-1)\theta + 2\delta$ which is at least 2π if and only if $r \geqslant 5$, $3\frac{2}{3}$, $4\frac{3}{5}$ and 3 in each case, respectively. This justifies the list given in the statement of the proposition.

Except in each of the two cases $G(5,3,4)$ and $G(3,5,3)$, where γ_2 is the shortest closed geodesic and has length 2π precisely, there is some excess of curvature in the link. This can be 'absorbed' into the interior of Y, by changing the metric. Thus, choose a different metric on Y (and $\widetilde{Y}$) so that Π is now a polytope in $\mathbb{H}^3 = M^3_{-1}$ (obtained, if you like, via the median construction on a regular solid in $\mathbb{H}^3$). The dimensions, or volume, of Π may be chosen to be sufficiently small that γ_0, γ_1 and γ_2 all have length still greater than 2π. Then $\mathrm{Lk}\widetilde{R}$ will still be CAT(1), and with the new metric $G(Y)$ will be negatively curved and $\widetilde{Y}$ a CAT(-1) polyhedron. $\square$

Proposition 3.3. *The polyhedron $\widetilde{Y}(5,3,4)$ contains isometrically embedded flat planes $\mathbb{E}^2$, while the polyhedron $\widetilde{Y}(3,5,3)$ does not.*

Proof. The polyhedron $\widetilde{Y}(5,3,4)$ is built from cuboctahedra which are glued together in pairs across each square face. Note that the 1-skeleton of the boundary of each cubeoctahedron contains paths of length 6 which bound regular Euclidean hexagons in the interior of the cuboctahedron. The flat plane which lies inside $\widetilde{Y}(5,3,4)$ is illustrated in Figure 6. It is built from the triangular faces of the cuboctahedra and hexagons of the type just described, and embeds in $\widetilde{Y}(5,3,4)$ in such a way that the link of each vertex maps onto the systole γ_2 inside Lk$\widetilde{R}$ for the appropriate vertex $\widetilde{R}$ above $R \in Y$.

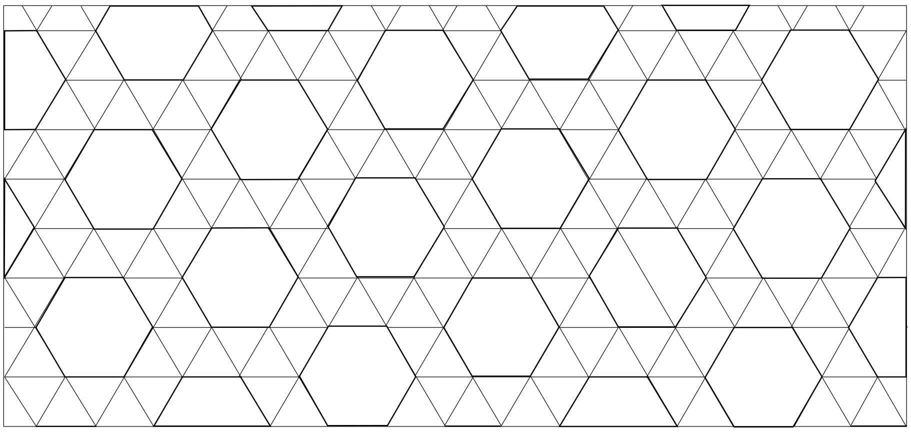


Figure 6: The flat plane in $\widetilde{Y}(5,3,4)$.

Suppose now that $\widetilde{Y}(3,5,3)$ contains an isometrically embedded flat plane E. Note that $\widetilde{Y}(3,5,3)$ is a union of icosadodecahedra glued along their triangular faces with three around each vertex. Necessarily the flat plane E intersects each icosadodecahedron Π either trivially or in a flat disk which is properly embedded with respect to the union Δ of triangular faces of Π (that is, the disk is transverse to Δ and its boundary lies in Δ). Thus $E \cap \Pi$ is either a pentagonal face of Π or a flat decagon which bisects Π and whose boundary lies in the 1-skeleton. It is possible to glue two pentagons and a single decagon around a vertex to have total angle exactly 2π (and this is realised in $\widetilde{Y}(3,5,3)$ along the systole γ_2), however it is impossible to build a whole flat plane with these tiles. Therefore $\widetilde{Y}(3,5,3)$ has no flat plane. $\square$

We define a further complex of groups $G(Z)$ which is contained in $G(Y)$ and consists of the triangle of groups spanned by P, R and C, as shown in Figure 7.

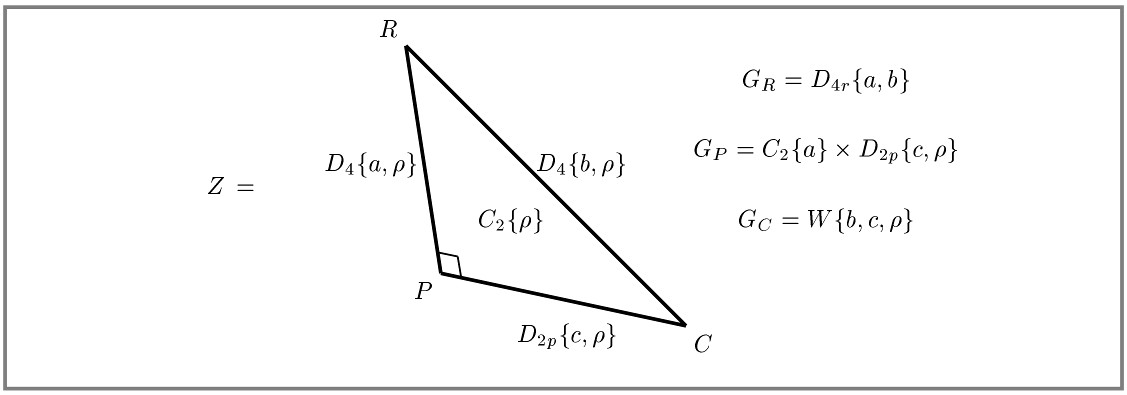


Figure 7: The complex of groups $G(Z)$.

Again, since the groups G_P, G_R and G_C generate $\pi_1(G(Y))$, this smaller complex of groups has the same fundamental group, and is developable whenever $G(Y)$ is. Moreover $\widetilde{Z}$ shall be identified with the subcomplex of $\widetilde{Y}$ lying above $Z \subset Y$.

Lemma 3.4. *Suppose that $G(Y)$ (and hence $G(Z)$) is a developable complex of groups. Then the $\widetilde{Z}$ is homotopy equivalent to (in fact a G-equivariant deformation retract of) $\widetilde{Y}$.*

Proof. Consider the quadrilateral $PQMC$ in Y. Let $f : PMQC \times [0,1] \to PMQC$ be the linear retraction of $PQMC$ onto its side PC which collapses the edge PQ down to P, and the edge MC down to C. We define a retraction $Y \to Z$ by extending f to all of Y via the cone construction at R and observe that this induces a G-equivariant homotopy retraction of $\widetilde{Y}$ onto $\widetilde{Z}$. □

Now suppose that Z is a Euclidean triangle with angles $\frac{\pi}{2}$ at P, $\frac{\pi}{r}$ at R, and $\psi = \pi(\frac{1}{2} - \frac{1}{r})$ at C. With these angles the polygon $G_R(PRC)$ is a regular Euclidean r-gon and $\widetilde{Z}$ is one of the Euclidean polygonal complexes studied by Ballman and Brin [1]. (We note that this metric on Z is quite different from the one that would be induced on it as a subspace of the Euclidean polytope Π.)

Proposition 3.5. *With the metric just described $G(Z)$ is non-positively curved, and $\widetilde{Z}$ is a CAT(0) polyhedron, if and only if*

$$\frac{1}{m} + \frac{1}{r} \leqslant \frac{1}{2}.$$

Moreover, if $\frac{1}{m} + \frac{1}{r} < \frac{1}{2}$, then the metric on Z may be chosen so that $G(Z)$ is negatively curved and $\widetilde{Z}$ is a CAT(-1) polyhedron.

Proof. By the choice of angles, the orbihedron links $\mathrm{Lk}\widetilde{P}$ and $\mathrm{Lk}\widetilde{R}$ are clearly CAT(1). The remaining link, $\mathrm{Lk}\widetilde{C}$, is a graph isomorphic to the 1-skeleton of the boundary of the platonic solid $P(m,p)$ in which every edge has length 2ψ. The shortest simple circuits in this graph are of length m. Thus $\mathrm{Lk}\widetilde{C}$ is CAT(1), and hence $G(Z)$ non-positively curved, if and only if $2m\psi \geqslant 2\pi$, or rather $m(\frac{1}{2} - \frac{1}{r}) \geqslant 1$ which is equivalent to the inequality stated in the Proposition.

When this inequality is strictly satisfied the CAT(1) link condition at C is still satisfied for some value of ψ strictly less than $\pi(\frac{1}{2} - \frac{1}{r})$. The triangle Z may then be given a metric of constant curvature $\chi = -1$ so as to have angles $\frac{\pi}{2}$, $\frac{\pi}{r}$, and ψ and still satisfy the link conditions. □

Comparison of Propositions 3.2 and 3.5 suggests a general improvement in curvature properties when going from an action on a 2-dimensional polyhedron to an action on a 3-dimensional one. For instance the 3-dimensional $\widetilde{Y}(r,m,p)$ is CAT(0) for $(r,m,p) = (5,3,4), (5,3,5)$ and $(3,5,3)$, while the 2-dimensional $\widetilde{Z}$ fails to be CAT(0) in each of these three cases. (In fact $\widetilde{Y}(5,3,5)$ admits an equivariant CAT(-1) metric). The significance of this observation is confirmed by the result of the following section (Proposition 4.2) which shows that if $G(r,m,p)$ (with $1/m + 1/p > 1/2$) acts properly discontinuously and cocompactly by isometries on any 2-dimensional CAT(0) (or CAT(-1)) polyhedron then the polyhedron might as well be $\widetilde{Z}$, i.e: $G(Z)$ is itself non-positively (or negatively) curved.

4 The fixed point argument

In this section we continue to refer to the notation introduced in the previous section in describing the group $G(r,m,p)$ by 2- and 3-dimensional complexes of groups $G(Z)$ and $G(Y)$ respectively, in the case $1/m + 1/p > 1/2$.

We shall need the following Lemma:

Lemma 4.1. *Let u, v, w be distinct non-collinear points in a 2-dimensional CAT(0) polyhedron Σ. Let u^+ and w^+ denote the points in $Lk(v)$ defined by the geodesic rays from v to u and w respectively. Then u^+ and w^+ are the endpoints of a unique geodesic path α in $Lk(v)$ and each point of α is defined by a geodesic ray from v to some point on the unique geodesic in Σ from u to w.*

Proof. For every $p \in \Sigma \setminus \{v\}$ the geodesic from v to p determines a unique point $p^+ \in \mathrm{Lk}(v)$, defining a continuous surjection $\Sigma \setminus \{v\} \to \mathrm{Lk}(v)$. Thus the geodesic path from u to w determines a path β from u^+ to w^+ in $\mathrm{Lk}(v)$. On the other hand, since $d(u^+, w^+) < \pi$ (u, v, w non-collinear), there is a unique geodesic path α from u^+ to w^+ of length $d(u^+, w^+)$.

Let λ, and μ, denote the geodesic paths in Σ from u, and w respectively, to v, and let σ_t denote the path in $\mathrm{Lk}(v)$ determined by the geodesic from $\lambda(t)$ to $\mu(t)$, for $0 \leqslant t < 1$, and let $\sigma_1 = \alpha$. Then σ is a homotopy of paths, relative to endpoints, from β to α.

Since Σ is 2-dimensional, $\mathrm{Lk}(v)$ is a graph and α is a simple path in $\mathrm{Lk}(v)$. It follows that β, which is homotopic to α, is a path whose image contains the image of α. Thus, for all $s \in [0, 1]$, $\alpha(s) = \beta(s')$ for some $s' \in [0, 1]$, and, by definition, $\beta(s') = k^+$ for some k on the geodesic path from u to w. □

Proposition 4.2. *Consider $G = G(r, m, p)$ with $1/m + 1/p > 1/2$ and $r \geqslant 2$. Suppose that the value r is chosen large enough that $G(Y)$ is a developable complex of groups with infinite fundamental group G. Then G acts geometrically on a 2-dimensional CAT(χ) polyhedral complex Σ with $\chi \leqslant 0$ if and only if $\frac{1}{r} + \frac{1}{m} \leqslant \frac{1}{2}$. Moreover, if $\frac{1}{r} + \frac{1}{m} < \frac{1}{2}$ then we may realise the action with $\chi < 0$.*

Remark. Note that if $G(Y)$ is non-positively curved then it is developable *and* G is an infinite group. The latter is true because, if it were finite, G would have a global fixed point by the Cartan Fixed Point Theorem (c.f. [8]). However no such global fixed point occurs in $\widetilde{Y}$.

Proof. The sufficiency of the condition, and the final statement both follow from Proposition 3.5. We consider here the necessity of the given condition.

Let Σ denote a 2-dimensional CAT(χ) polyhedral complex on which the group $G = \pi_1(G(Z))$ acts geometrically. Recall that $G(Z)$ is the triangle of groups spanned by vertices P, R and C in Y. We denote by F_R, F_P and F_C respectively the fixed point sets in Σ of the finite groups G_R, G_P and G_C. Since the action is properly discontinuous and any two of the groups G_R, G_P and G_C generate the whole of $G(Y)$, which we suppose to be infinite, the fixed sets F_R, F_P and F_C must be mutually disjoint subsets of Σ. Moreover, since each of G_R, G_P and G_C fixes a single point in $\widetilde{Y}$ it is its own normaliser in G. Consequently, since G acts co-compactly on Σ, each of F_R, F_P, and F_C is a compact set.

Thus, by compactness, we may choose points $v_R \in F_R$, $v_P \in F_P$ and $v_C \in F_C$ so as to minimise the perimeter $d(v_R, v_P) + d(v_P, v_C) + d(v_R, v_C)$ of the triangle $\Delta(v_R, v_P, v_C)$. Let δ_a denote the geodesic path in Σ from v_R to v_P. Similarly define δ_b from v_R to v_C, and δ_c from v_P to v_C. (See Figure 8).

Let u, v, w denote (in no particular order) the vertices of the minimal perimeter triangle Δ just introduced. The vertex v is the common endpoint of a pair of sides δ_x (from v to u) and

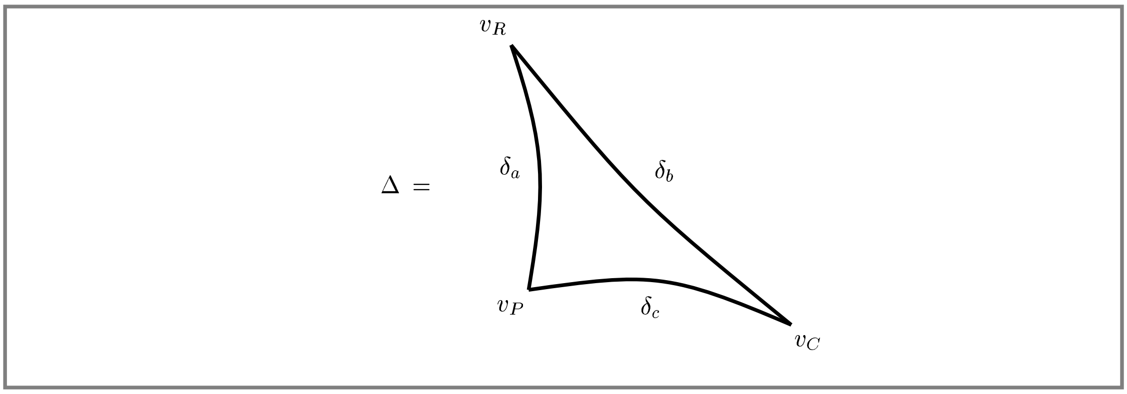


Figure 8: The geodesic triangle Δ in Σ.

δ_y (from v to w), for some $x, y \in \{a, b, c\}$. Note that the side δ_x of Δ is fixed by the subgroup generated by x and ρ, and that the stabiliser G_v of the vertex v is the group generated by x, y and ρ. (Compare Figure 8 with Figure 7).

Consider now the action of the elements $x, y \in G_v$ on $\mathrm{Lk}(v)$. In particular, let α be the unique geodesic path in $\mathrm{Lk}(v)$ between u^+ and w^+, as in Lemma 4.1. Let θ denote $\ell(\alpha)$ which is just the angle between two sides of Δ at v, and is strictly less than π. We note that the element $\rho \in G_v$ obviously fixes α pointwise, since it fixes both u^+ and w^+.

Suppose both x and y fix some point p of α. By Lemma 4.1, $p = k^+$ is defined by the geodesic from v to a point k on the opposite side of Δ. It follows that both x and y, and hence all elements of G_v, fix a small initial segment of the geodesic from v to k. (Clearly ρ fixes the whole path between v and k). Choose $v' \neq v$ on this segment (close to v). Applying the comparison axiom to triangles $\Delta(u, v, k)$ and $\Delta(w, v, k)$ and using the fact that k lies on the side of Δ between u and w, one sees that $\Delta' = \Delta(u, v', w)$ has strictly smaller perimeter than Δ, contradicting the choice of Δ.

Therefore we may suppose that no point of α is fixed by both x and y. Generally, $Fix(x) \cap \alpha$ is a closed subinterval $[u^+, p]$ of α, for some $p \in \alpha$, since it is closed and convex in α and contains the endpoint u^+. Similarly $Fix(y) \cap \alpha = [q, w^+]$, for some $q \in \alpha$. In the present case, these subintervals are disjoint. In particular, we have a nontrivial subinterval $[p, q]$ of α such that $[p, q] \cap x([p, q]) = p$ and $[p, q] \cap y([p, q]) = q$. The translates of $[p, q]$ by the elements $1, x, xy, xyx, xyxy$, and so on, form a simple closed curve in the graph $\mathrm{Lk}(v)$ of length $\lambda \leqslant 2n\theta$, where n is the smallest positive integer such that $(xy)^n \in \langle \rho \rangle$. Since $\mathrm{Lk}(v)$ is a CAT(1) link, it now follows that $\theta \geqslant \pi/n$.

Inspecting the vertex groups individually, we have $n = r, m$ and 2 for the vertices v_R, v_C and v_P respectively. Since the angle sum of a triangle is at most π in a CAT(0) space, and strictly less in CAT(-1), we now have $\pi/r + \pi/m + \pi/2 \leqslant \pi$ (resp. $< \pi$) for the triangle Δ in Σ. The conclusion of the Proposition follows immediately. □

Proof of Main Theorem. The Theorem stated in the Introduction now follows from the information contained in Propositions 3.2 and 3.5 together with the above Proposition 4.2. □

5 Appendix

5.1 Property FA and one-endedness

We recall the following property of a group introduced by Serre in [15]. A group G is said to have *property FA* if every action of g on a tree has a global fixed point.

Lemma 5.1. *Let $r, p \geqslant 1$ and $m \geqslant 2$. Then the group $G(r, m, p)$ has Serre's property FA. In particular $G(r, m, p)$ is either finite or 1-ended.*

Proof. This follows because $G(r, m, p)$ is a quotient of a triangle group Δ and, in fact, all triangle groups $\Delta(i, j, k)$, with $i, j, k \geqslant 2$ and finite, have property FA. To see this, suppose that $\Delta = \Delta(i, j, k)$ acts on a tree T. Then each finite subgroup fixes a point. The group Δ is generated by three reflections a, b, c such that any two reflections generate a finite dihedral group: thus $< a, b > \cong D_{2i}$, $< b, c > \cong D_{2j}$ and $< a, c > \cong D_{2k}$. Let I, J, K denote points in T which are fixed by these three dihedral groups respectively. Let $M \in T$ denote the median of I, J, K (that is the unique point which lies in $[I, J] \cap [J, K] \cap [K, I]$). Clearly each of the elements a, b and c must fix M, and so M is a global fixed point under the action of Δ on T. □

5.2 The 16 exceptional cases of Proposition 2.1(2).

$\backslash$ p m $\backslash$	3	4	5	6	7	8	9
5	120	320	1320	*virt.* $\mathbb{Z}^3$	∞	$\mathbb{Z}^2 \subset G$	∞
6	216	*virt.* $\mathbb{Z}^3$	∞				
7	2	$\mathbb{Z}^2 \subset G$					
8	1344						
9	6840						
10	∞						
11	∞						

Table 2: Order of the group $G(2, m, p)$ for different values of m, p. Only groups which appear in the list of 16 exceptionals are indicated (the rest are CAT(0) and infinite).

Recall that the group $G(r, m, p)$ is isomorphic to the group of combinatorial automorphisms of the tiled surface (or orbifold) $S(r, m, p)$ defined in Section 2. In order to give a little of the flavour of these groups we describe briefly the surfaces $S(r, m, p)$ associated to some of the 16 exceptional cases of Proposition 2.1(2). These arise in the case where $r = 2$. The orders of these 16 groups are shown in Table 2 (computations were made in GAP). Of those which are infinite, the groups $G(2, 5, 6)$ and $G(2, 6, 4)$ are virtually $\mathbb{Z}^3$, and the groups $G(2, 5, 8)$ and $G(2, 7, 4)$ contain subgroups isomorphic to $\mathbb{Z}^2$ (see below). It would be interesting to know whether any or all of the remaining examples are Gromov hyperbolic groups.

Consider the pentagonal tiled surfaces of Figure 9. The first of these is a sphere with 8 boundary components, each of edge length 6 (a cube with its corners cut off). Forming the quotient of this surface by the antipodal map (of the 2-sphere) and identifying opposite points on each boundary component yields the surface $S(2,5,3)$ which is nonorientable of Euler characteristic -3. Figure 9(ii) shows a torus with four holes. The surface $S(2,5,4)$ is obtained by doubling this surface along its boundary, giving an orientable surface of Euler characteristic -8. These tiled surfaces have finite automorphism groups of order equal to 10 times the number of pentagonal faces: 120 and 320 respectively.

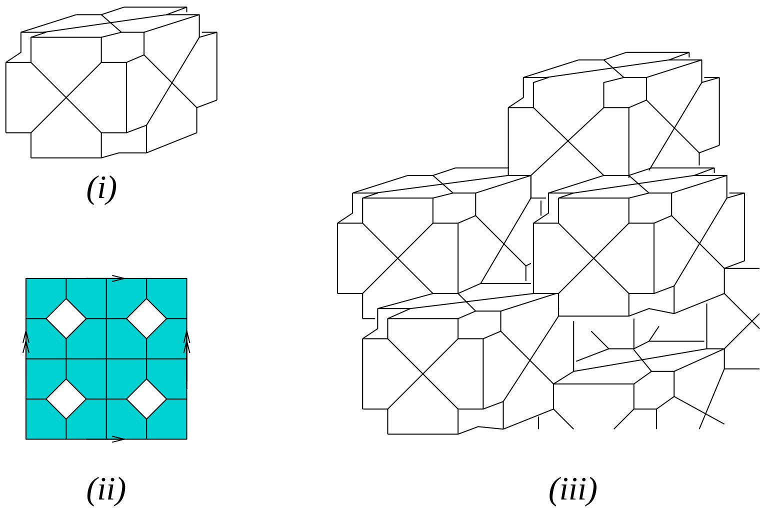


Figure 9: Pentagonal tiled surfaces.

Infinitely many copies of the surface of Figure 9 (i) may be arranged in Euclidean 3-space and glued together at their boundaries to form an infinite tiled surface which is embedded in $\mathbb{R}^3$, as shown in Figure 9 (iii). This surface is isomorphic to $S(2,5,6)$ and clearly has an automorphism group which is virtually $\mathbb{Z}^3$. A similar description is possible for the surface $S(2,5,6)$, however here the "punctured cube" is replaced by an infinitely punctured planar surface with hyperbolic symmetry group (a discrete subgroup of $Isom(\mathbb{H}^2)$).

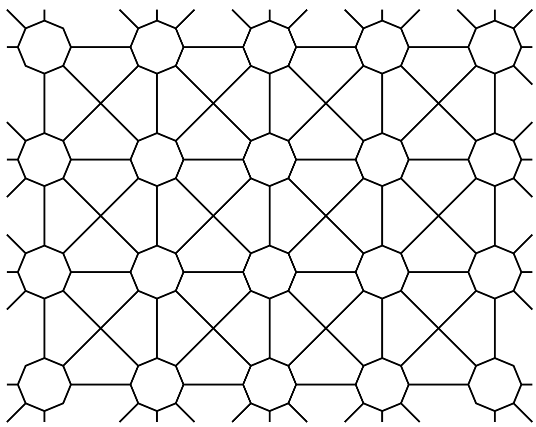


Figure 10: Planar subsurface of $G(2,5,8)$.

The surface $S(2,5,8)$ contains subsurfaces isomorphic to the punctured Euclidean plane illustrated in Figure 10, and may be built by gluing together copies of this planar surface in an appropriate way. In particular, since the complex is totally regular, any isometry of the planar subsurface induces an isometry of the whole surface, and so $G(2,5,8)$ contains a subgroup $\mathbb{Z}^2$.

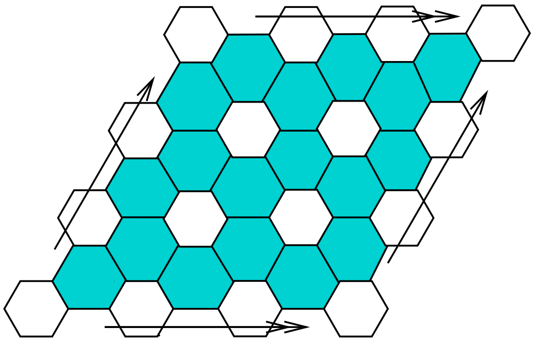


Figure 11: The finite surface of $S(2,6,3)$.

Figure 11 shows a hexagonal tiling of a torus with nine holes. Identifying opposite points in each boundary component yields a nonorientable surface of Euler characteristic -9 which is isomorphic to $S(2,6,3)$. (The automorphism group has order $18 \times 12 = 216$).

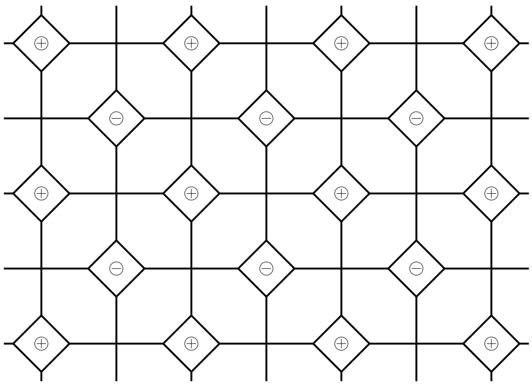


Figure 12: Planar subsurface of $S(2,6,4)$.

The infinitely punctured tiled planar surface of Figure 12 may be doubled along the set of boundary components labelled $\oplus$ and those labelled $\ominus$. These two doublings may be thought of as reflections which generate an infinite dihedral group. The complex thus developed may be embedded in Euclidean 3-space and admits a automorphism group which is virtually $\mathbb{Z}^3$. This surface is $S(2,6,4)$.

The surface $S(2,7,4)$ may be built from gluing together copies of the punctured flat shown in Figure 13 in an appropriate manner. As with $S(2,5,8)$ the automorphism group of this surface contains $\mathbb{Z}^2$ subgroups.

5.3 Proof of Propositions 2.1 and 2.2.

We are given a triple of whole numbers (r,m,p) where $p \geqslant 3$ and r,m satisfy

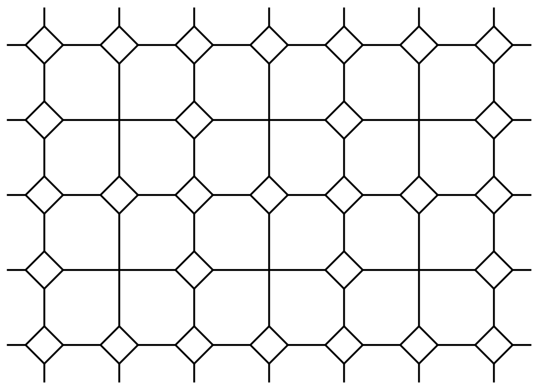


Figure 13: Planar subsurface of $G(2, 7, 4)$.

$$\text{(I)} \qquad \frac{1}{m} + \frac{1}{2r} < \frac{1}{2}.$$

The piecewise Euclidean complex of groups $G(X)$ is non-positively curved (or $\widetilde{X}$ a CAT(0) polyhedron) if and only if the link L illustrated in Figure 2, with $\theta = (\frac{1}{2} - \frac{1}{m})\pi$ and $\phi = (\frac{1}{2} - \frac{1}{p})\pi$, is CAT(1). There are three candidates for a shortest closed geodesic in L, as illustrated in Figure 14.

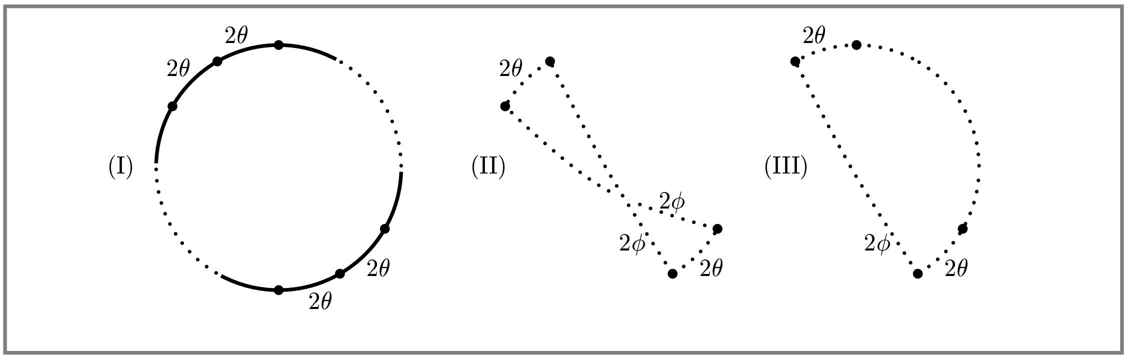


Figure 14: Candidates for the shortest circuit in L.

The first of these has length $4r\theta$, which is $> 2\pi$ by virtue of inequality (I). The other two have lengths $4\theta + 4\phi$ and $2r\theta + 2\phi$ respectively. Thus L is CAT(1) if and only if both of the following inequalities hold

$$\text{(II)} \qquad \frac{1}{m} + \frac{1}{p} \leqslant \frac{1}{2},$$

$$\text{(III)} \qquad \frac{r}{m} + \frac{1}{p} \leqslant \frac{r-1}{2}.$$

Let (II)* and (III)* respectively denote the corresponding strict inequalities. Then $G(X)$ may be given a metric of strictly negative curvature, and $\widetilde{X}$ a CAT(-1) metric, if and only if both (II)* and (III)* hold.

We note, firstly, that case (1) of Proposition 2.1 arises precisely when inequality (II) fails. Also, the cases where (II) holds but (II)* fails are precisely the triples $(r,3,6)$, $(r,4,4)$ and $(r,6,3)$ which are listed in Proposition 2.2 (the extra conditions on r stated in the proposition coming from inequality (I) above).

We may now suppose, for the proof of Proposition 2.1, that (II) holds and, for the proof of Proposition 2.2, that (II)* holds. The following Lemma reduces our considerations even further, to the case $r = 2$.

Lemma 5.2. *We suppose that (I) holds, and that $r \neq 2$. (In general $r \geqslant 2$). Then*

$$\begin{aligned} \textit{(II)} &\implies \textit{(III)} \quad \textit{and} \\ \textit{(II)}^* &\implies \textit{(III)}^* \quad . \end{aligned}$$

Proof. Inequality (III) may be rewritten

$$\text{(III)}' \qquad \frac{1}{m} + \frac{1}{p} \leqslant \frac{(r-1)}{2} - \frac{(r-1)}{m},$$

the strict version of (III)$'$ being equivalent to (III)*. Thus it suffices to show (for both statements in the Lemma) that $(r-1)(\frac{1}{2} - \frac{1}{m}) \geqslant \frac{1}{2}$.

If $m \geqslant 4$, then $(r-1)(\frac{1}{2} - \frac{1}{m}) \geqslant \frac{r-1}{4} \geqslant \frac{1}{2}$, since $r \geqslant 3$. Otherwise $m = 3$ and inequality (I) implies that $r \geqslant 4$. But in this case $(r-1)(\frac{1}{2} - \frac{1}{m}) = \frac{r-1}{6} \geqslant \frac{1}{2}$, completing the proof. $\square$

This leaves us merely to consider the inequalities (III) and (III)* in the case $r = 2$. But, with $r = 2$, (III) holds in all cases except the 16 listed in case (2) of Proposition 2.1, and (III)* holds in all but these cases and the cases $(r,m,p) = (2,6,6)$ and $(2,8,4)$ listed in Proposition 2.2.

Finally, we must demonstrate the existence of isometrically embedded flat planes in the various borderline cases listed in Proposition 2.2. But, in the case $G(X)$ is nonpositively curved, one can always develop a chequerboard tiling of the hyperbolic or Euclidean plane by m-gons and p-gons as a subcomplex of $\widetilde{X}$. More precisely we take the geodesically convex subcomplex of $\widetilde{X}$ corresponding to the subcomplex of groups $G'(X)$ in which $G'_M = G_M$, $G'_{MR} = C_2\langle b\rangle$, $G'_{MQR} = G'_{PQR} = \langle 1\rangle$, $G'_Q = G'_{MQ} = G'_{PQ} = C_2\langle c\rangle$, $G'_P = D_{2p}\langle c,\rho\rangle$, $G'_{PR} = C_2\langle\rho\rangle$, and $G'_R = C_2\langle\rho\rangle \times C_2\langle b\rangle$. This tiled plane is most often hyperbolic, but is Euclidean precisely in the cases referred to in Proposition 2.2. $\square$

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