

## REPRESENTATIONS OF THE BRAID GROUP BY AUTOMORPHISMS OF GROUPS, INVARIANTS OF LINKS, AND GARSIDE GROUPS

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From a group  $H$  and  $h \in H$ , we define a representation  $\rho : B_n \rightarrow \text{Aut}(H^{*n})$ , where  $B_n$  denotes the braid group on  $n$  strands, and  $H^{*n}$  denotes the free product of  $n$  copies of  $H$ . We call  $\rho$  the Artin type representation associated to the pair  $(H, h)$ . Here we study various aspects of such representations.

Firstly, we associate to each braid  $\beta$  a group  $\Gamma_{(H,h)}(\beta)$  and prove that the operator  $\Gamma_{(H,h)}$  determines a group invariant of oriented links. We then give a topological construction of the Artin type representations and of the link invariant  $\Gamma_{(H,h)}$ , and we prove that the Artin type representations are faithful if and only if  $h$  is nontrivial. The last part of the paper is devoted to the study of some semidirect products  $H^{*n} \rtimes_{\rho} B_n$ , where  $\rho : B_n \rightarrow \text{Aut}(H^{*n})$  is an Artin type representation. In particular, we show that  $H^{*n} \rtimes_{\rho} B_n$  is a Garside group if  $H$  is a Garside group and  $h$  is a Garside element of  $H$ .

### 1. Introduction

Throughout the paper, we shall denote by  $B_n$  the braid group on  $n$  strands, and by  $\sigma_1, \dots, \sigma_{n-1}$  the standard generators of  $B_n$ .

Let  $H$  be a group and fix  $h \in H$ . Take  $n$  copies  $H_1, \dots, H_n$  of  $H$  and consider the group  $H^{*n} = H_1 * \dots * H_n$ . We denote by  $\phi_i : H \rightarrow H_i$  the natural isomorphism and we write  $h_i = \phi_i(h) \in H_i$ , for all  $i = 1, \dots, n$ . For  $k = 1, \dots, n - 1$ , let  $\tau_k : H^{*n} \rightarrow H^{*n}$  be the automorphism determined by

$$\tau_k : \begin{cases} \phi_k(y) \mapsto h_k^{-1} \phi_{k+1}(y) h_k, \\ \phi_{k+1}(y) \mapsto h_k \phi_k(y) h_k^{-1}, \\ \phi_j(y) \mapsto \phi_j(y) & \text{if } j \neq k, k+1 \end{cases}$$

for  $y \in H$ . One can easily show the following.

**Proposition 1.1.** *The mapping  $\sigma_k \mapsto \tau_k$ ,  $k = 1, \dots, n - 1$ , determines a representation  $\rho : B_n \rightarrow \text{Aut}(H^{*n})$ .*

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*Proof.* This involves checking, case by case, that the usual braid group relations are satisfied by the automorphisms  $\tau_k$ . For example, both  $\tau_k \tau_{k+1} \tau_k$  and  $\tau_{k+1} \tau_k \tau_{k+1}$  map  $\phi_k(y)$  to  $h_k^{-1} h_{k+1}^{-1} \phi_{k+2}(y) h_{k+1} h_k$ ,  $\phi_{k+1}(y)$  to  $h_k^{-1} h_{k+1} \phi_{k+1}(y) h_{k+1}^{-1} h_k$ , etc. Similarly, one checks that  $\tau_k \tau_j = \tau_j \tau_k$  if  $k < j - 1$ . We leave the details to the reader.  $\square$

**Definition 1.2.** The representation of [Proposition 1.1](#) shall be called the *Artin type representation of  $B_n$  associated to the pair  $(H, h)$* .

The special case where  $h$  is taken to be the identity,  $h = \text{Id}_H$ , gives a representation of  $B_n$  by permutations of the free factors of  $H^{*n}$ . This representation has image the full symmetric group  $S_n$  and kernel the pure braid group. All other Artin type representations will be shown to be faithful (see [Proposition 4.1](#)).

If  $H = \mathbb{Z}$  and  $h = 1$  (a generator of  $\mathbb{Z}$  in the additive notation), then  $H^{*n} = F_n$  is the free group of rank  $n$  and  $\rho$  is the classical representation introduced by Artin [[1925](#); [1947](#)]. Another example which appears in the literature is the case where  $H = \mathbb{Z}$  and  $h$  is an arbitrary nonzero integer. This case was introduced by Wada [[1992](#)] in his construction of group invariants of links. Sections [2](#) and [3](#) of the present paper are inspired by [[Wada 1992](#)].

Our purpose in this paper is to study different aspects of the Artin type representations.

**Definition 1.3.** Let  $\rho : B_n \rightarrow \text{Aut}(H^{*n})$  be the Artin type representation associated to a pair  $(H, h)$ . Let  $\beta \in B_n$ . Then we denote by  $\Gamma(\beta) = \Gamma_{(H,h)}(\beta)$  the quotient of  $H^{*n}$  by the relations

$$g = \rho(\beta)g, \quad g \in H^{*n}.$$

For a braid  $\beta$ , we denote by  $\hat{\beta}$  the oriented link (or more precisely the equivalence class of oriented links) represented by the closed braid of  $\beta$  as defined in [[Birman 1974](#)]. Given two braids  $\beta_1$  and  $\beta_2$  (not necessarily with the same number of strands), we prove in [Section 2](#) that  $\Gamma(\beta_1) \simeq \Gamma(\beta_2)$  if  $\hat{\beta}_1 = \hat{\beta}_2$ . This allows us to define a group invariant of oriented links,  $\Gamma_{(H,h)}$ , by setting  $\Gamma_{(H,h)}(L)$  to be the group  $\Gamma_{(H,h)}(\beta)$  for any braid  $\beta$  such that  $L = \hat{\beta}$ . Note that, in the case  $H = \mathbb{Z}$  and  $h = 1$ , the invariant  $\Gamma_{(\mathbb{Z},1)}$  computes the link group, namely  $\Gamma_{(\mathbb{Z},1)}(L) \cong \pi_1(\mathcal{S}^3 \setminus L)$  for any link  $L$  in  $\mathcal{S}^3$ .

The goal of [Section 3](#) is to give topological constructions of the Artin type representations and of the groups  $\Gamma_{(H,h)}(\beta)$ , for  $\beta \in B_n$ . If  $H = \mathbb{Z}$  and  $h$  is a nonzero integer, then our constructions coincide with Wada's constructions [[1992](#), Section 3]. In fact, our constructions are straightforward extensions of Wada's constructions to all Artin type representations.

In [Section 4](#), we prove that Artin type representations are faithful whenever  $h$  is chosen nontrivial ([Proposition 4.1](#)). If  $h$  has infinite order, then the Artin type

representation  $\rho : B_n \rightarrow \text{Aut}(H^{*n})$  contains the classical Artin representation and, therefore, is faithful by [Artin 1925; 1947]. So, Proposition 4.1 is mostly of interest in the case where  $h$  has finite order. In fact the proof may be easily reduced to the case  $H = \mathbb{Z}/k\mathbb{Z}$  and  $h = 1$ , however we will not need to use any such reduction, as our method applies just as easily in all cases. We note also that the case where  $H$  is cyclic of order 2 follows (by somewhat different methods) from [Crisp and Paris 2002, Section 2.3]. The proof of Proposition 4.1 is inspired by the proof of [Shpilrain 2001, Theorem A], and it is based on Dehornoy's work [1994; 1997a] on orderings of the braid group.

The remaining sections (Sections 5 and 6) are dedicated to the study of semidirect products  $H^{*n} \rtimes_{\rho} B_n$ , where  $\rho : B_n \rightarrow \text{Aut}(H^{*n})$  is the Artin type representation associated to a pair  $(H, h)$ .

If  $H = \mathbb{Z}$  and  $h = 1$ , then  $H^{*n} \rtimes_{\rho} B_n$  is the Artin group  $A(B_n)$  associated to the Coxeter graph  $B_n$  (not to be confused with the braid group  $B_n$ , which is itself an Artin group, of type  $A_{n-1}$ ). This result is implicit in [Lambropoulou 1994; Crisp 1999], and is described explicitly in [Crisp and Paris 2002]. The group  $A(B_n)$  is well-understood. In particular, solutions to the word and conjugacy problems in this group are known [Deligne 1972; Brieskorn and Saito 1972], it is torsion free [Brieskorn 1973; Deligne 1972], its center is an infinite cyclic group [Deligne 1972; Brieskorn and Saito 1972], it is biautomatic [Charney 1992; 1995], and it has an explicit finite dimensional classifying space [Deligne 1972; Bestvina 1999].

A natural next step is to understand the groups  $H^{*n} \rtimes_{\rho} B_n$  in the case where  $\rho$  is a Wada representation (of type 4), namely, when  $H = \mathbb{Z}$  and  $h \in \mathbb{Z} \setminus \{0\}$ . One can readily establish that, for these representations, the group  $H^{*n} \rtimes_{\rho} B_n$  fails to be an Artin group unless  $h = \pm 1$ . It turns out, however, that these groups do have quite a lot in common with Artin groups: like the Artin groups, they belong to a family of groups known as *Garside groups*.

Briefly, a *Garside group* is a group  $G$  which admits a left invariant lattice order and contains a so-called *Garside element*, a positive element  $\Delta$  whose positive divisors generate  $G$  and such that conjugation by  $\Delta$  leaves the lattice structure invariant (there are also conditions placed on the positive cone of  $G$ , that it be a finitely generated atomic monoid; see Section 5 for details). The notion of a Garside group was introduced in [Dehornoy and Paris 1999] in a slightly restricted sense, and in [Dehornoy 2002] in the larger sense in which is now generally used. The theory of Garside groups is largely inspired by [Garside 1969], which treated the case of braid groups, and [Brieskorn and Saito 1972], which generalized Garside's work to Artin groups. The Artin groups of spherical (or finite) type which include, notably, the braid groups as well as the groups  $A(B_n)$  mentioned above, are motivating examples. Other interesting examples of Garside groups include all

torus link groups (see [Picantin 2003]) and some generalized braid groups associated to finite complex reflection groups.

Garside groups have many attractive properties. Solutions to the word and conjugacy problems in these groups are known [Dehornoy 2002; Picantin 2001b; Franco and González-Meneses 2003], they are torsion free [Dehornoy 1998], they admit canonical decompositions as iterated direct products of “irreducible” components, and the center of each component is an infinite cyclic group [Picantin 2001a], they are biautomatic [Dehornoy 2002], and they admit finite dimensional classifying spaces [Dehornoy and Lafont 2003; Charney et al. 2004]. Another important property of the Garside groups is that there exist criteria in terms of presentations to detect them [Dehornoy and Paris 1999; Dehornoy 2002].

In Section 6, we prove that, if  $H$  is a Garside group,  $h$  a Garside element of  $H$ , and  $\rho$  the Artin type representation associated to  $(H, h)$ , then  $H^{*n} \rtimes_{\rho} B_n$  is also a Garside group (Theorem 6.1). This result applies in particular to the case  $H = \mathbb{Z}$  and  $h \in \mathbb{Z} \setminus \{0\}$ , but also applies, for example, to the case where  $H$  is another braid group, say  $H = B_l$ , and  $h = \Delta^k$  is a nontrivial power of the fundamental element of  $B_l$ .

The proof of Theorem 6.1 is based on a necessary and sufficient criterion for a group to be Garside explained in Section 5. This criterion rests largely on the “coherence” condition of [Dehornoy and Paris 1999] and is essentially a variation on [Dehornoy 2002, Proposition 6.14]. Our version differs from Dehornoy’s [2002] in that it is not algorithmic. In particular, we do not give any method for finding a Garside element. However, our Criterion 5.9 is relatively easy to apply once one has an appropriate presentation and an expression for a Garside element to hand.

Finally, in the Appendix we answer a question posed by Shpilrain [2001] in his study of Wada’s representations.

**Definition 1.4.** Let  $G$  be a group. Two representations  $\rho, \rho' : B_n \rightarrow \text{Aut}(G)$  are called *equivalent* if there exist automorphisms  $\phi : G \rightarrow G$  and  $\mu : B_n \rightarrow B_n$  such that  $\rho'(\mu(\beta)) = \phi^{-1} \circ \rho(\beta) \circ \phi$  for all  $\beta \in B_n$ .

**Remark.** If two representations  $\rho, \rho' : B_n \rightarrow \text{Aut}(G)$  are equivalent, then the groups  $G \rtimes_{\rho} B_n$  and  $G \rtimes_{\rho'} B_n$  are isomorphic.

Shpilrain’s question was simply to give a classification of Wada’s representations up to equivalence. This classification is given in Proposition A.3.

## 2. Link invariants

Let  $H$  be a group,  $h \in H$ , and  $\rho : B_n \rightarrow \text{Aut}(H^{*n})$  be the Artin type representation associated to  $(H, h)$ . Recall that the group  $H^{*n}$  is defined as  $H^{*n} = H_1 * \cdots * H_n$ ,

where group isomorphisms  $\phi_i : H_i \rightarrow H$  are given for  $i = 1, 2, \dots, n$ . The goal of this section is to prove the following.

**Proposition 2.1.** *Let  $n, m \in \mathbb{N}$ , and let  $\beta_1 \in B_n$  and  $\beta_2 \in B_m$ . If  $\hat{\beta}_1 = \hat{\beta}_2$ , then  $\Gamma_{(H,h)}(\beta_1) \simeq \Gamma_{(H,h)}(\beta_2)$ .*

**Definition 2.2** (Link invariant). Let  $L$  be an oriented link. We set  $\Gamma_{(H,h)}(L) := \Gamma_{(H,h)}(\beta)$ , where  $\beta$  is any braid (on any number of strings) such that  $L = \hat{\beta}$ . By [Proposition 2.1](#),  $\Gamma_{(H,h)}$  is a well-defined group invariant of oriented links.

*Proof of [Proposition 2.1](#).* Let  $n \in \mathbb{N}$  and let  $\beta \in B_n$ . We write  $\Gamma$  for  $\Gamma_{(H,h)}$ . By Markov's theorem [[Birman 1974](#), Theorem 2.3], it suffices to show that

I assumed [Birm] was the same as [Bir]

- (1)  $\Gamma(\alpha^{-1}\beta\alpha) \simeq \Gamma(\beta)$  for all  $\alpha \in B_n$ ,
- (2)  $\Gamma(\beta\sigma_n) \simeq \Gamma(\beta)$ , and
- (3)  $\Gamma(\beta\sigma_n^{-1}) \simeq \Gamma(\beta)$ ,

where  $\beta\sigma_n$  and  $\beta\sigma_n^{-1}$  are viewed as braids on  $n + 1$  strands.

Note that, if  $\beta \in B_n$  and  $n \leq m$ , then the action of  $\beta$  via  $\rho$  on  $H^{*m}$  agrees with the action via  $\rho$  on  $H^{*n} < H^{*m}$ , and is trivial on the free factors  $H_{n+1}, \dots, H_m$ . We suppress  $\rho$  from our notation, writing simply  $\beta(g)$  to mean  $\rho(\beta)g$ , for any  $\beta \in B_n$  and  $g \in H^{*m}$ . This also amounts to writing  $\sigma_k$  instead of  $\tau_k$ .

We now prove conditions (1), (2) and (3) above.

(1) For  $\beta \in B_n$ , the group  $\Gamma(\beta)$  is defined as the quotient of  $H^{*n}$  by the relations  $g = \beta(g)$  for all  $g \in H^{*n}$ . Since, for  $\alpha \in B_n$ , the relation  $g = \alpha^{-1}\beta\alpha(g)$  is equivalent to the relation  $\alpha(g) = \beta(\alpha(g))$ , and  $\alpha$  is an automorphism of  $H^{*n}$ , it is clear that  $\Gamma(\alpha^{-1}\beta\alpha)$  is defined by the same set of relations as  $\Gamma(\beta)$ .

(2) The group  $\Gamma(\beta\sigma_n)$  may be defined as the quotient of  $H^{*(n+1)}$  by the family of relations  $R(i, x) : \phi_i(x) = \beta\sigma_n(\phi_i(x))$  for  $i = 1, 2, \dots, n + 1$  and  $x \in H$ . Note that  $\sigma_n(\phi_{n+1}(x)) = h_n\phi_n(x)h_n^{-1}$ . Therefore the relation  $R(n + 1, x)$  is equivalent to the relation  $R'(n + 1, x) : \phi_{n+1}(x) = \beta(h_n\phi_n(x)h_n^{-1})$ , where the right hand side is actually an element of  $H^{*n}$ . In particular  $\Gamma(\beta\sigma_n)$  is generated by the image of  $H^{*n}$ . Also,

$$\beta\sigma_n(\phi_n(x)) = \beta(h_n^{-1}\phi_{n+1}(x)h_n) = \beta(h_n^{-1})\phi_{n+1}(x)\beta(h_n).$$

So, in view of  $R'(n + 1, x)$ , the relation  $R(n, x)$  is now equivalent to the relation  $R'(n, x) : \phi_n(x) = \beta(\phi_n(x))$ . Finally, since  $\sigma_n(\phi_i(x)) = \phi_i(x)$  for all  $i < n$ , the remaining relations  $R(i, x)$  are equivalent to  $R'(i, x) : \phi_i(x) = \beta(\phi_i(x))$  for all  $i = 1, 2, \dots, n - 1$ , and all  $x \in H$ . It now follows that  $\Gamma(\beta\sigma_n) \simeq \Gamma(\beta)$ .

(3) Observe that  $\Gamma(\beta^{-1}) \simeq \Gamma(\beta)$ , since the relation  $g = \beta(g)$  is equivalent to  $\beta^{-1}(g) = g$ , for all  $g \in H^{*n}$ . Then

$$\begin{aligned} \Gamma(\beta\sigma_n^{-1}) &\simeq \Gamma(\sigma_n\beta^{-1}) \\ &\simeq \Gamma(\beta^{-1}\sigma_n) \quad \text{by the proof of (1),} \\ &\simeq \Gamma(\beta^{-1}) \quad \text{by the proof of (2),} \\ &\simeq \Gamma(\beta). \end{aligned} \quad \square$$

### 3. Topological construction of the link invariants

Let  $X$  be a CW-complex, let  $P_0 \in X$  be a basepoint, and let  $\alpha : [0, 1] \rightarrow X$  be a loop based on  $P_0$ . In this section we give a topological realization of the Artin type representation of  $B_n$  associated to the pair  $(H, h) = (\pi_1(X, P_0), [\alpha])$ , and we deduce a topological construction of the link invariant  $\Gamma_{(H, h)}$  of the previous section.

Let  $\mathbf{D} = \mathbf{D}(\frac{n+1}{2}, \frac{n+1}{2})$  denote the disk in  $\mathbb{C}$  centered at  $\frac{n+1}{2}$  of radius  $\frac{n+1}{2}$ . Now, we construct a space  $Y$  obtained from  $\mathbf{D}$  by making  $n$  holes in  $\mathbf{D}$  and gluing a copy of  $X$  into each hole by identifying the circular boundary of the hole to the loop  $\alpha$  in  $X$ .

Choose some small  $\varepsilon > 0$  (we require only that  $\varepsilon < \frac{1}{8}$ ). Let

$$Y' = \mathbf{D} \setminus \left( \bigcup_{k=1}^n \mathring{\mathbf{D}}(k, \varepsilon) \right),$$

where  $\mathring{\mathbf{D}}(k, \varepsilon)$  denotes the open disk centered at  $k$  of radius  $\varepsilon$ . Take  $n$  copies  $X_1, \dots, X_n$  of  $X$ , denote by  $f_k : X \rightarrow X_k$  the natural homeomorphism, and write  $\alpha_k = f_k \circ \alpha$  for all  $k = 1, \dots, n$ . Then

$$Y = \left( Y' \sqcup \left( \bigsqcup_{k=1}^n X_k \right) \right) / \sim,$$

where  $\sim$  is the identification defined by

$$\alpha_k(t) \sim k + \varepsilon e^{2\sqrt{-1}\pi t}, \quad k = 1, \dots, n, \quad t \in [0, 1].$$

Finally, choose a basepoint  $Q_0 \in \partial \mathbf{D}$  for  $Y$ . The following result is a direct consequence of the above construction.

**Lemma 3.1.** *Let  $H = \pi_1(X, P_0)$ , and let  $H_1, \dots, H_n$  be  $n$  copies of  $H$ . Then  $\pi_1(Y, Q_0) \simeq H_1 * \dots * H_n$ .*

We now show that the braid group  $B_n$  acts on  $Y$  up to isotopy relative to the boundary of  $\mathbf{D}$  in such a way that the induced action on  $\pi_1(Y)$  is the Artin type

representation associated to  $(H, h)$ , where  $h$  is the element of  $H = \pi_1(X, P_0)$  represented by  $\alpha$ .

Let  $\xi \in \mathbb{C}$  and  $0 < r < R$ . Define the *half Dehn twist*  $T = T(\xi, r, R)$  by

$$T(\xi + \rho e^{\sqrt{-1}\theta}) = \begin{cases} \xi + \rho e^{\sqrt{-1}(\theta-\pi)} & \text{if } 0 \leq \rho \leq r, \\ \xi + \rho e^{\sqrt{-1}(\theta-t\pi)} & \text{if } r \leq \rho \leq R \text{ and } t = \frac{R-\rho}{R-r}, \\ \xi + \rho e^{\sqrt{-1}\theta} & \text{if } \rho \geq R \end{cases}$$

(see Figure 1).

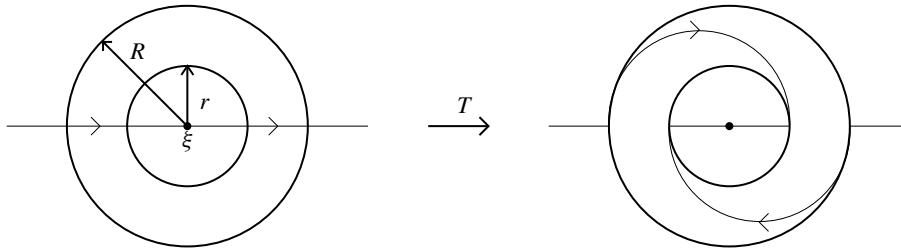


Figure 1. A half Dehn twist.

Let  $T_k^D : D \rightarrow D$  be the homeomorphism defined by

$$T_k^D = T(k, \varepsilon, 2\varepsilon)^{-3} \circ T(k + 1, \varepsilon, 2\varepsilon)^{-1} \circ T(k + \frac{1}{2}, \frac{1}{2} + \varepsilon, \frac{1}{2} + 2\varepsilon).$$

Note that  $T_k^D$  leaves invariant the set  $\bigcup_{j=1}^n D(j, \varepsilon)$ , and therefore restricts to a homeomorphism  $T'_k : Y' \rightarrow Y'$ . See Figure 2.

One can verify (with a little effort) that  $T'_k T'_{k+1} T'_k$  is isotopic to  $T'_{k+1} T'_k T'_{k+1}$  relative to  $\partial Y'$  for  $k = 1, \dots, n - 2$ , and that  $T'_k T'_l$  is isotopic to  $T'_l T'_k$  relative to

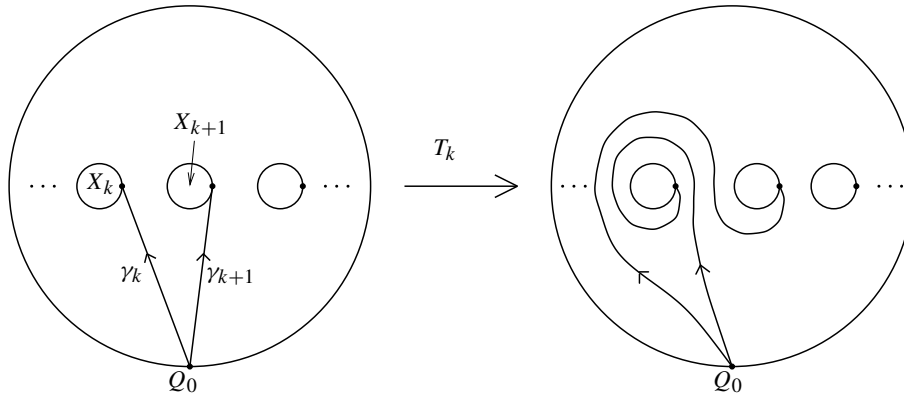


Figure 2. The homeomorphism  $T'_k : Y' \rightarrow Y'$ .

$\partial Y'$  for  $|k-l| \geq 2$ . Moreover,  $T'_k$  fixes  $\partial \mathbf{D}$  and transforms the rest of  $\partial Y'$  as follows:

$$T'_k(j + \varepsilon e^{\sqrt{-1}\theta}) = \begin{cases} j + \varepsilon e^{\sqrt{-1}\theta} & \text{if } j \neq k, k+1, \\ k+1 + \varepsilon e^{\sqrt{-1}\theta} & \text{if } j = k, \\ k + \varepsilon e^{\sqrt{-1}\theta} & \text{if } j = k+1. \end{cases}$$

Therefore,  $T'_k$  extends to a homeomorphism  $T_k : Y \rightarrow Y$  by setting, for all  $x \in X$ ,

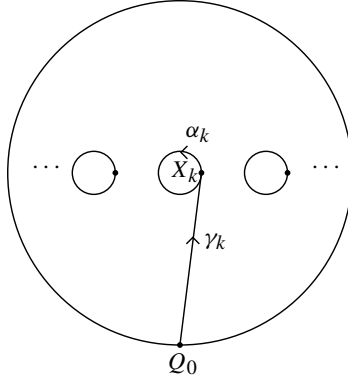
$$T_k(f_j(x)) = \begin{cases} f_j(x) & \text{if } j \neq k, k+1, \\ f_{k+1}(x) & \text{if } j = k, \\ f_k(x) & \text{if } j = k+1. \end{cases}$$

The homeomorphism  $T_k$  is the identity on  $\partial \mathbf{D}$ ,  $T_k T_{k+1} T_k$  is isotopic to  $T_{k+1} T_k T_{k+1}$  relatively to  $\partial \mathbf{D}$  for  $k = 1, \dots, n-2$ , and  $T_k T_l$  is isotopic to  $T_l T_k$  relatively to  $\partial \mathbf{D}$  for  $|k-l| \geq 2$ .

These observations show that  $T_k$  determines an automorphism  $\tau_k : \pi_1(Y, Q_0) \rightarrow \pi_1(Y, Q_0)$ . Moreover,

$$\begin{aligned} \tau_k \tau_{k+1} \tau_k &= \tau_{k+1} \tau_k \tau_{k+1} & \text{for } k = 1, \dots, n-2, \\ \tau_k \tau_l &= \tau_l \tau_k & \text{for } |k-l| \geq 2. \end{aligned}$$

Thus the mapping  $\sigma_k \rightarrow \tau_k$  determines a representation  $\rho : B_n \rightarrow \text{Aut}(\pi_1(Y, Q_0))$ .



**Figure 3.** The path  $\gamma_k$ .

Set  $Q_0 = \frac{n+1}{2} - \sqrt{-1} \frac{n+1}{2}$ . Let  $\gamma_k : [0, 1] \rightarrow Y$  be the path from  $Q_0$  to  $f_k(P_0)$  shown in [Figure 3](#). We identify  $\pi_1(Y, Q_0)$  with  $H^{*n} = H_1 * \dots * H_n$  in such a way that the  $k$ -th embedding  $\phi_k : H = \pi_1(X, P_0) \rightarrow H_k \subset H^{*n}$  is defined by

$$\phi_k([\beta]) = [\gamma_k f_k(\beta) \gamma_k^{-1}].$$

With this assumption, one can easily show the following.

**Proposition 3.2.** *The representation  $\rho : B_n \rightarrow \text{Aut}(\pi_1(Y, Q_0))$  described above coincides with the Artin type representation of  $B_n$  associated to  $(H, h)$ , where  $H = \pi_1(X, P_0)$  and  $h$  is the element of  $H$  represented by  $\alpha$ .*

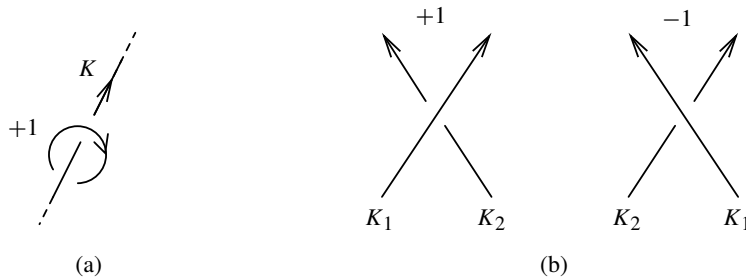
*Proof.* It suffices to observe, with the aid of [Figure 2](#), that, for all  $k = 1, \dots, n-1$  and all loops  $\beta$  at  $P_0$  in  $X$ ,

- (i)  $T_k(\gamma_k f_k(\beta) \gamma_k^{-1})$  is homotopic to  $\gamma_k \alpha_k^{-1} \gamma_k^{-1} \gamma_{k+1} f_{k+1}(\beta) \gamma_{k+1}^{-1} \gamma_k \alpha_k \gamma_k^{-1}$ ;
- (ii)  $T_k(\gamma_{k+1} f_{k+1}(\beta) \gamma_{k+1}^{-1})$  is homotopic to  $\gamma_k \alpha_k f_k(\beta) \alpha_k^{-1} \gamma_k^{-1}$ ; and
- (iii)  $T_k(\gamma_j f_j(\beta) \gamma_j^{-1})$  is homotopic to  $\gamma_j f_j(\beta) \gamma_j^{-1}$ , for all  $j \neq k, k+1$ .  $\square$

We now introduce some standard notions and facts concerning framings of links and linking numbers. We refer the reader to [\[Rolfsen 1990\]](#), or any similar introductory text on knot theory, for further details.

Consider an oriented  $m$ -component link  $L = K_1 \cup \dots \cup K_m$  in  $S^3$ . The knot  $K_i$  is an embedding  $K_i : S^1 \rightarrow S^3$ , and  $K_i(S^1) \cap K_j(S^1) = \emptyset$  for  $i \neq j$ . Define a *tubular neighborhood* of  $K_i$  to be an embedding  $T_i : D^2 \times S^1 \rightarrow S^3$  such that  $T_i(0, \xi) = K_i(\xi)$  for all  $\xi \in S^1$ . Here,  $D^2$  denotes the disk centered at 0 of radius 1 in  $\mathbb{C}$ . A *framing* of  $L$  is a collection  $\{T_i : D^2 \times S^1 \rightarrow S^3\}_{i=1}^m$  of embeddings such that  $T_i$  is a tubular neighborhood of  $K_i$ , for  $i = 1, \dots, m$ , and  $T_i(D^2 \times S^1) \cap T_j(D^2 \times S^1) = \emptyset$  for  $i \neq j$ . The *longitude* of the component  $K_i$  is the (oriented) embedding  $\lambda_i : S^1 \rightarrow S^3$  such that  $\lambda_i(\xi) = T_i(1, \xi)$  for all  $\xi \in S^1$ . The tubular neighborhood of the framing of each component  $K_i$  is determined up to isotopy by the homology class of its longitude  $\lambda_i$  in the knot complement  $S^3 \setminus K_i$ .

Given an oriented knot  $K$ , we identify  $H_1(K) := H_1(S^3 \setminus K)$  with  $\mathbb{Z}$  in such a way that  $1 \in \mathbb{Z}$  is represented by the 1-cycle depicted in [Figure 4\(a\)](#). Let  $K_1, K_2$  denote disjoint oriented knots in  $S^3$ . One defines the *linking number*  $\text{lk}(K_1, K_2) \in \mathbb{Z}$  to be the class  $[K_1] \in H_1(K_2) = \mathbb{Z}$ . The linking number  $\text{lk}(K_1, K_2)$  may be measured from any regular projection of the link  $K_1 \cup K_2$  by counting with sign the crossings where  $K_1$  passes over  $K_2$ , as indicated in [Figure 4\(b\)](#). (Equally one



**Figure 4.** Sign conventions.

may choose to count undercrossings with the appropriate sign, and one quickly sees that  $\text{lk}(K_1, K_2) = \text{lk}(K_2, K_1)$ .

**Notation** (Preferred framing). Let  $L = K_1 \cup \cdots \cup K_m$  be an  $m$ -component oriented link in  $\mathcal{S}^3$ . Up to isotopy, there is a unique framing in which the longitude  $\lambda_i$  for each component  $K_i$  satisfies the condition

$$\sum_{j=1}^m \text{lk}(\lambda_i, K_j) = 0.$$

Note that, for  $j \neq i$ ,  $\text{lk}(\lambda_i, K_j) = \text{lk}(K_i, K_j)$  and is determined by the oriented link  $L$ . We shall refer to the above framing as the *preferred framing* of  $L$ .

We now wish to associate to an oriented link  $L$  the space  $\Omega(L, X)$  obtained by performing a ‘generalized’ surgery on the link  $L$  according to the preferred framing just described. More precisely, let  $L = K_1 \cup \cdots \cup K_m$  and let  $\{T_i : \mathbf{D}^2 \times \mathcal{S}^1 \rightarrow \mathcal{S}^3\}_{i=1}^m$  be the preferred framing. Let  $\mathring{T}_i$  denote the interior of  $T_i(\mathbf{D}^2 \times \mathcal{S}^1)$  for  $i = 1, \dots, m$ , and set

$$\Omega'(L) = \mathcal{S}^3 \setminus \left( \bigcup_{i=1}^m \mathring{T}_i \right).$$

Take  $m$  copies  $X_1, \dots, X_m$  of  $X$ , denote by  $f_i : X \rightarrow X_i$  the natural homeomorphism, and write  $\alpha_i = f_i \circ \alpha$ . Then

$$\Omega(L, X) = \left( \Omega'(L) \sqcup \left( \bigsqcup_{i=1}^m (X_i \times \mathcal{S}^1) \right) \right) / \sim,$$

where  $\sim$  is the identification defined by putting

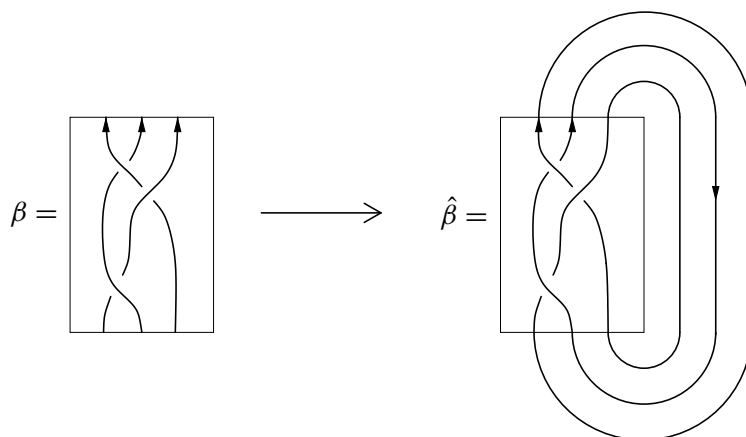
$$(\alpha_i(t), \eta) \sim T_i(e^{2\sqrt{-1}\pi t}, \eta), \quad i = 1, \dots, m, \quad t \in [0, 1], \quad \eta \in \mathcal{S}^1.$$

The following proposition yields a second proof of the fact that  $\Gamma_{(H,h)}$  is a link invariant for any finitely generated group  $H$  and any element  $h \in H$ .

**Proposition 3.3.** *Let  $\beta$  be a braid, and let  $\hat{\beta}$  denote the closed braid of  $\beta$ . Let  $X$  be a CW-complex with basepoint  $P_0$  and let  $\alpha$  be a loop in  $X$ . Then  $\pi_1(\Omega(\hat{\beta}, X))$  is isomorphic to  $\Gamma_{(H,h)}(\beta)$ , where  $H = \pi_1(X, P_0)$  and  $h$  is the element of  $H$  represented by  $\alpha$ .*

*Proof.* We first remind the reader of the standard construction of the closed braid  $\hat{\beta}$  from a braid  $\beta$  [Birman 1974]. The notation used to describe this construction will be needed for the completion of the proof. Firstly, decompose  $\mathcal{S}^3$  as follows: let  $T_1, T_2$  be two copies of the solid torus  $\mathbf{D} \times \mathcal{S}^1$  and write

$$\mathcal{S}^3 = T_1 \bigcup_{\kappa: \partial T_1 \rightarrow \partial T_2} T_2,$$



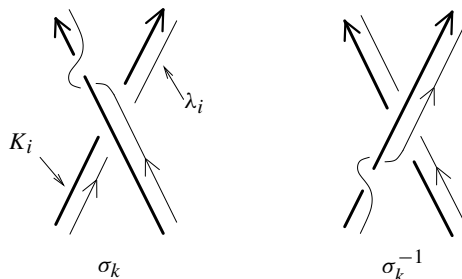
**Figure 5.** Braid closure.

where the identifying map  $\kappa$  is a homeomorphism carrying  $\partial D$  to  $S^1$  and  $S^1$  to  $\partial D$ . Let  $g$  denote the inclusion of  $T_1$  in  $S^3$ , and let  $f : D \times [0, 1] \rightarrow T_1 = D \times S^1$  be the identification map defined by

$$f(p, t) = (p, e^{2\sqrt{-1}\pi t}), \quad \text{for } p \in D \text{ and } t \in [0, 1].$$

The closed braid  $\hat{\beta}$  is the oriented link which is induced by composing the braid  $\beta : \{1, \dots, n\} \times [0, 1] \rightarrow D \times [0, 1]$  with the map  $g \circ f : D \times [0, 1] \rightarrow S^3$ . The orientation on  $\hat{\beta}$  is naturally induced from a choice of orientation of the interval  $[0, 1]$ .

Given a standard projection of a braid  $\beta$  we may describe a projection of the closed braid  $\hat{\beta}$  with the same number of crossings, as indicated in Figure 5. We now produce a framing  $\Lambda$  of  $\hat{\beta}$  by choosing a longitude  $\lambda_i$  for each component  $K_i$  of  $\hat{\beta}$  whose projections are as indicated in Figure 6 in the vicinity of a crossing, and otherwise parallel to the link projection. It is easily enough verified, by counting overcrossings, that this framing is exactly the preferred framing of  $\hat{\beta}$ .



**Figure 6.** Choosing a framing for  $\hat{\beta} = K_1 \cup \dots \cup K_m$ .

Write  $\beta = \sigma_{k_1}^{\varepsilon_1} \sigma_{k_2}^{\varepsilon_2} \dots \sigma_{k_r}^{\varepsilon_r}$  and define  $T_\beta^{\mathbf{D}} : \mathbf{D} \rightarrow \mathbf{D}$  as the composition of the homeomorphisms  $(T_{k_j}^{\mathbf{D}})^{\varepsilon_j}$  for  $j = 1, \dots, r$ . Similarly, define  $T_\beta = T_{k_1}^{\varepsilon_1} T_{k_2}^{\varepsilon_2} \dots T_{k_r}^{\varepsilon_r} : Y \rightarrow Y$ . For  $j = 1, \dots, n$ , denote by  $b_j$  the point  $j + \varepsilon$  on  $\partial \mathbf{D}(j, \varepsilon)$ . This is the point on  $\partial Y'$  to which the basepoint of  $X_j$  is attached when forming  $Y$ . Since  $T_\beta^{\mathbf{D}}$  is isotopic to  $\text{Id}_{\mathbf{D}}$  relative to  $\partial \mathbf{D}$ , there is a homeomorphism  $U : \mathbf{D} \times [0, 1] \rightarrow \mathbf{D} \times [0, 1]$  such that  $U(x, 0) = (x, 0)$ ,  $U(x, 1) = (T_\beta^{\mathbf{D}}(x), 1)$ , for all  $x \in \mathbf{D}$ , and  $U$  fixes  $\partial \mathbf{D} \times [0, 1]$  pointwise. Moreover, by construction,  $U$  carries

$$\left( \bigsqcup_{j=1}^n \mathbf{D}(j, \varepsilon) \right) \times [0, 1]$$

to a tubular neighborhood of (a representative of) the braid  $\beta$ , and  $g \circ f \circ U$  carries the arcs  $\{b_j \times [0, 1] : j = 1, \dots, n\}$  to a framing of  $\widehat{\beta}$  equivalent to that described in [Figure 6](#), namely the preferred framing. Consequently the space  $\Omega(\widehat{\beta}, X)$  is homeomorphic to  $T'_1 \cup T_2$  where

$$T'_1 = Y \times [0, 1] / ((y, 0) \sim (T_\beta(y), 1)).$$

We therefore have  $\pi_1(T'_1) \cong H^{*n} \star \langle t \rangle / (txt^{-1} = \rho(\beta)x \text{ for } x \in H^{*n})$ , an HNN-extension. Attaching  $T_2$  to  $T'_1$  has the effect of simply killing the stable letter  $t$ . Consequently

$$\pi_1(\Omega(\widehat{\beta}, X)) \cong H^{*n} / (x = \rho(\beta)x \text{ for } x \in H^{*n}) = \Gamma_{(H, h)}(\beta). \quad \square$$

#### 4. Faithfulness

Recall that, for any group  $H$  and for  $n \in \mathbb{N}$ , we write  $H^{*n}$  for the free product  $H_1 * \dots * H_n$ , where each free factor  $H_i$  is isomorphic to  $H$  by an isomorphism  $\phi_i : H \rightarrow H_i$ . The aim of this section is to prove the following.

**Proposition 4.1.** *Let  $\rho : B_n \rightarrow \text{Aut}(H^{*n})$  be the Artin type representation of  $B_n$  associated to the pair  $(H, h)$  where  $H$  is a group and  $h \in H$ .*

- (i) *If  $h \neq \text{Id}_H$ , then  $\rho$  is faithful.*
- (ii) *If  $h = \text{Id}_H$ , then  $\ker(\rho)$  is the pure braid group and  $B_n / \ker(\rho) \cong S_n$ , the symmetric group, acts by permutations of the free factors of  $H^{*n}$  (respecting the isomorphisms  $\{\phi_1, \dots, \phi_n\}$ ).*

**Remark.** Part (ii) of this proposition requires no proof but is included here for completeness. We concern ourselves below with the case of  $h$  nontrivial.

As pointed out in the introduction, the proof of [Proposition 4.1\(i\)](#) is strongly inspired by the proof of [[Shpilrain 2001](#), Theorem A], and its main ingredient is the following:

**Proposition 4.2** [Dehornoy 1994; 1997a]. *Let  $B_{[2,n]}$  denote the subgroup of  $B_n$  generated by  $\sigma_2, \dots, \sigma_{n-1}$  (namely the braid group on the second through  $n$ th strings). Let  $\beta \in B_n$ . Then either*

- (1)  $\beta \in B_{[2,n]}$ , or
- (2) one of  $\beta$  or  $\beta^{-1}$  can be written as  $\alpha_0 \sigma_1 \alpha_1 \sigma_1 \alpha_2 \dots \sigma_1 \alpha_l$ , where  $l \geq 1$  and  $\alpha_0, \dots, \alpha_l \in B_{[2,n]}$ .

The following lemma is preliminary to the proof of Proposition 4.1. We first fix a nontrivial  $h \in H$ , and write  $h_i = \phi_i(h)$  for  $i = 1, \dots, n$ .

**Lemma 4.3.** *Let  $K = H_2 * \dots * H_n$ . Let  $u \in H^{*n}$  such that the normal form of  $u$  with respect to the decomposition  $H^{*n} = H_1 * K$  starts with  $h_1^{-1}$  and ends with  $h_1$ .*

- (1) *The normal form of  $\rho(\sigma_1)(u)$  with respect to the decomposition  $H^{*n} = H_1 * K$  also starts with  $h_1^{-1}$  and ends with  $h_1$ .*
- (2) *Let  $k \in \{2, \dots, n-1\}$  and  $\varepsilon \in \{\pm 1\}$ . The normal form of  $\rho(\sigma_k^\varepsilon)(u)$  with respect to the decomposition  $H^{*n} = H_1 * K$  also starts with  $h_1^{-1}$  and ends with  $h_1$ .*

*Proof.* Let  $v \in H_1 * H_2$ . Suppose that the normal form of  $v$  is

$$v = \phi_1(x_1) \phi_2(y_1) \dots \phi_1(x_l) \phi_2(y_l),$$

where  $x_1, \dots, x_l, y_1, \dots, y_{l-1} \in H \setminus \{\text{Id}\}$ , and  $y_l \in H$ . Then

$$\rho(\sigma_1)(v) = h_1^{-1} \cdot \phi_2(x_1) \cdot h_1^2 \phi_1(y_1) h_1^{-2} \cdot \dots \cdot \phi_2(x_l) \cdot h_1^2 \phi_1(y_l) h_1^{-1};$$

thus the normal form of  $\rho(\sigma_1)(v)$  starts with  $h_1^{-1}$ .

Similarly, if the normal form of  $v$  is

$$v = \phi_2(y_1) \phi_1(x_1) \dots \phi_2(y_l) \phi_1(x_l),$$

where  $x_1, \dots, x_l, y_2, \dots, y_l \in H \setminus \{\text{Id}\}$  and  $y_1 \in H$ , then the normal form of  $\rho(\sigma_1)(v)$  ends with  $h_1$ .

Now, write

$$u = v_0 w_1 v_1 \dots w_l v_l,$$

where  $v_i \in (H_1 * H_2) \setminus \{\text{Id}\}$  and  $w_j \in (H_3 * \dots * H_n) \setminus \{\text{Id}\}$ , and  $l \geq 0$ . The hypothesis that  $u$  starts with  $h_1^{-1}$  implies that  $v_0$  starts with  $h_1^{-1}$ , and the hypothesis that  $u$  ends with  $h_1$  implies that  $v_l$  ends with  $h_1$ . Both groups,  $H_1 * H_2$  and  $H_3 * \dots * H_n$ , are invariant by  $\rho(\sigma_1)$ , and  $\rho(\sigma_1)$  is the identity on  $H_3 * \dots * H_n$ . So,

$$\rho(\sigma_1)(u) = \rho(\sigma_1)(v_0) \cdot w_1 \cdot \rho(\sigma_1)(v_1) \cdot \dots \cdot w_l \cdot \rho(\sigma_1)(v_l).$$

By the observations above,  $\rho(\sigma_1)(v_0)$  starts with  $h_1^{-1}$  and  $\rho(\sigma_1)(v_l)$  ends with  $h_1$ ; thus  $\rho(\sigma_1)(u)$  starts with  $h_1^{-1}$  and ends with  $h_1$ .

Let  $k \in \{2, \dots, n-1\}$  and  $\varepsilon \in \{\pm 1\}$ . Write

$$u = h_1^{-1} w_1 v_1 \dots v_{l-1} w_l h_1,$$

where  $v_1, \dots, v_{l-1} \in H_1 \setminus \{\text{Id}\}$  and  $w_1, \dots, w_l \in K \setminus \{\text{Id}\}$ . Both groups,  $H_1$  and  $K$ , are invariant by  $\rho(\sigma_k^\varepsilon)$ , and  $\rho(\sigma_k^\varepsilon)$  is the identity on  $H_1$ . So

$$\rho(\sigma_k^\varepsilon)(u) = h_1^{-1} \cdot \rho(\sigma_k^\varepsilon)(w_1) \cdot v_1 \cdots v_{l-1} \cdot \rho(\sigma_k^\varepsilon)(w_l) \cdot h_1;$$

thus the normal form of  $\rho(\sigma_k^\varepsilon)(u)$  starts with  $h_1^{-1}$  and ends with  $h_1$ .  $\square$

*Proof of Proposition 4.1(i).* We argue by induction on  $n$ . Assume  $n = 2$ . We have

$$\begin{aligned} \rho(\sigma_1^{2l})(h_1) &= (h_2 h_1)^{-1} h_1 (h_2 h_1)^l \neq h_1, \quad \text{for } l \in \mathbb{Z} \setminus \{0\} \\ \rho(\sigma_1^{2l+1})(h_1) &= (h_2 h_1)^{-1} h_1^{-1} h_2 h_1 (h_2 h_1)^l \neq h_1, \quad \text{for } l \in \mathbb{Z}; \end{aligned}$$

thus the representation  $\rho : B_2 \rightarrow \text{Aut}(H_1 * H_2)$  is faithful.

Now, assume  $n \geq 3$ . Let  $\beta \in B_n \setminus \{\text{Id}\}$ . By Proposition 4.2, either  $\beta \in B_{[2,n]}$ , or one of  $\beta$  or  $\beta^{-1}$  is written  $\alpha_0 \sigma_1 \dots \sigma_1 \alpha_l$ , where  $l \geq 1$  and  $\alpha_0, \dots, \alpha_l \in B_{[2,n]}$ .

Suppose  $\beta \in B_{[2,n]}$ . By induction,  $\rho(\beta)$  acts nontrivially on  $K = H_2 * \dots * H_n$ ; thus  $\rho(\beta)$  acts nontrivially on  $H^{*n} = H_1 * K$ .

Suppose  $\beta = \alpha_0 \sigma_1 \dots \sigma_1 \alpha_l$ , where  $l \geq 1$  and  $\alpha_0, \dots, \alpha_l \in B_{[2,n]}$ . Let

$$u = \rho(\sigma_1 \alpha_l)(h_1) = \rho(\sigma_1)(h_1) = h_1^{-1} h_2 h_1.$$

By Lemma 4.3, the normal form of  $\rho(\alpha_0 \sigma_1 \dots \sigma_1 \alpha_{l-1})(u) = \rho(\beta)(h_1)$  starts with  $h_1^{-1}$  and ends with  $h_1$ . In particular,  $\rho(\beta)(h_1) \neq h_1$ ; thus  $\rho(\beta) \neq \text{Id}$ .

Finally, suppose  $\beta^{-1} = \alpha_0 \sigma_1 \dots \sigma_1 \alpha_l$ , where  $l \geq 1$  and  $\alpha_0, \dots, \alpha_l \in B_{[2,n]}$ . By the previous case,  $\rho(\beta^{-1}) \neq \text{Id}$ ; thus  $\rho(\beta) \neq \text{Id}$ .  $\square$

## 5. Garside groups

In this section we give a brief presentation of the definition and salient properties of a Garside group, and then establish the necessary and sufficient criteria for a group to be a Garside group which we shall use in the subsequent section. Our Criterion 5.9 is essentially a variation on [Dehornoy 2002, Proposition 2.1]. The theory of Garside groups, as developed in [Dehornoy and Paris 1999; Dehornoy 1997b; 2002], provides the most natural general setting for the combinatorial arguments contained in Garside's original treatment [1969] of the braid groups, and its generalization to Artin groups in [Brieskorn and Saito 1972].

**Definition 5.1.** Let  $M$  be an arbitrary monoid. We say that  $M$  is *atomic* if there exists a function  $v : M \rightarrow \mathbb{N}$  such that

- $v(a) = 0$  if and only if  $a = 1$ ;
- $v(ab) \geq v(a) + v(b)$  for all  $a, b \in M$ .

Such a function  $\nu : M \rightarrow \mathbb{N}$  is called a *norm* on  $M$ . An element  $a \in M$  is called an *atom* if it is indecomposable, namely, if  $a = bc$  then either  $b = 1$  or  $c = 1$ .

We note that any generating set of  $M$  contains the set of all atoms. In particular,  $M$  is finitely generated if and only if it has only finitely many atoms. For details see [Dehornoy and Paris 1999].

Given that a monoid  $M$  is atomic, we may define left and right invariant partial orders  $\leq_L$  and  $\leq_R$  on  $M$  as follows:

- set  $a \leq_L b$  if there exists  $c \in M$  such that  $ac = b$ ;
- set  $a \leq_R b$  if there exists  $c \in M$  such that  $ca = b$ .

We shall call these the *left* and *right divisibility orders* on  $M$ .

**Definition 5.2.** A *Garside monoid* is a monoid  $M$  such that

- (i)  $M$  is atomic and finitely generated;
- (ii)  $M$  is (left and right) cancellative, i.e.  $abc = ab'c$  implies  $b = b'$ ;
- (iii)  $(M, \leq_L)$  and  $(M, \leq_R)$  are lattices;
- (iv) there exists an element  $\Delta \in M$ , which we call a *Garside element*, such that
  - (a) the set  $L(\Delta) := \{x \in M : x \leq_L \Delta\}$  generates  $M$ , and
  - (b) the sets  $L(\Delta)$  and  $R(\Delta) := \{x \in M : x \leq_R \Delta\}$  are equal.

**Definition 5.3.** For any monoid  $M$  one can define the group  $G(M)$  which is presented by the generating set  $M$  and relations  $ab = c$  whenever  $ab = c$  in  $M$ . There is an obvious canonical homomorphism  $M \rightarrow G(M)$ . This homomorphism is not injective in general. The group  $G(M)$  is known as the *group of fractions* of  $M$ . Define a *Garside group* to be the group of fractions of a Garside monoid.

**Remark.** (1) A Garside monoid  $M$  satisfies Öre's conditions; thus the canonical homomorphism  $M \rightarrow G(M)$  is injective. Moreover the partial orders  $\leq_L$  and  $\leq_R$  extend respectively to left- and right-invariant lattice orders on  $G(M)$  with positive cone  $M$ .

(2) A Garside element is never unique. For example, if  $\Delta$  is a Garside element, then  $\Delta^k$  is also a Garside element for all  $k \geq 1$  [Dehornoy 2002, Lemma 2.2].

(3) Elsewhere in the literature the condition that  $M$  is finitely generated is often incorporated into condition (iv) of the definition by saying that the set  $L(\Delta)$  is finite. It seems more natural to state this condition separately. Note that, if  $M$  is finitely generated and atomic, then  $L(a) = \{x \in M : x \leq_L a\}$  is finite for all  $a \in M$ .

We now introduce some terminology needed in order to state [Criterion 5.9](#).

For a finite set  $S$ , we denote by  $S^*$  the free monoid on  $S$ . The elements of  $S^*$  are called *words* on  $S$ . The empty word is denoted by  $\epsilon$ . Let  $\equiv$  be a congruence

relation on  $S^*$ , and let  $M = (S^* / \equiv)$ . For  $w \in S^*$ , we denote by  $\bar{w}$  the element of  $M$  represented by  $w$ , and we call  $w$  an *expression* of  $\bar{w}$ .

**Definition 5.4.** A *complement* is a function  $f : S \times S \rightarrow S^*$  such that  $f(x, x) = \epsilon$  for all  $x \in S$ . To a complement  $f : S \times S \rightarrow S^*$  we associate the two monoids

$$M_L^f = \langle S \mid xf(x, y) = yf(y, x) \text{ for } x, y \in S \rangle^+,$$

$$M_R^f = \langle S \mid f(y, x)x = f(x, y)y \text{ for } x, y \in S \rangle^+.$$

For  $u, v \in S^*$ , we write  $u \equiv_L^f v$  if  $u$  and  $v$  are expressions of the same element of  $M_L^f$ , and we write  $u \equiv_R^f v$  if  $u$  and  $v$  are expressions of the same element of  $M_R^f$ .

**Definition 5.5.** A word  $w$  in  $(S \cup S^{-1})^*$  is *f-reversible on the left in one step* to a word  $w'$  if  $w'$  is obtained from  $w$  by replacing some subword  $x^{-1}y$  (with  $x, y \in S$ ) by the corresponding word  $f(x, y)f(y, x)^{-1}$ . Let  $p \geq 0$ . We say that  $w$  is *f-reversible on the left in  $p$  steps* to a word  $w'$  if there exists a sequence  $w_0 = w, w_1, \dots, w_p = w'$  in  $(S \cup S^{-1})^*$  such that  $w_{i-1}$  is *f-reversible on the left in one step* to  $w_i$  for all  $i = 1, \dots, p$ . The property “ $w$  is *f-reversible on the left* to  $w'$ ” is denoted by  $w \xrightarrow{f}_L w'$ .

We define *f-reversibility on the right* in a similar way, replacing subwords  $yx^{-1}$  (with  $x, y \in S$ ) by the corresponding words  $f(x, y)^{-1}f(y, x)$ . The property “ $w$  is *f-reversible on the right* to  $w'$ ” is denoted by  $w \xrightarrow{f}_R w'$ .

It is shown in [Deh3] that a reversing process is confluent, namely:

**Proposition 5.6** [Dehornoy 1997b, Lemma 1]. *Let  $f : S \times S \rightarrow S^*$  be a complement, and let  $w \in (S \cup S^{-1})^*$ . Suppose that the word  $w$  is *f-reversible on the left in  $p$  steps* to a word  $uv^{-1}$ , with  $u, v \in S^*$ . Then any sequence of left *f-reversing transformations* starting from  $w$  leads in  $p$  steps to  $uv^{-1}$ .*

reversing or reversible?

**Definition 5.7.** Let  $f : S \times S \rightarrow S^*$  be a complement and let  $u, v \in S^*$ . Assume that there exist  $u', v' \in S^*$  such that  $u^{-1}v \xrightarrow{f}_L u'(v')^{-1}$ . By Proposition 5.6,  $u'$  and  $v'$  are unique (if they exist). Then we write  $u' = C_L^f(u, v)$  and  $v' = C_L^f(v, u)$ . One has, by [Dehornoy 1997b, Lemma 2],

$$uC_L^f(u, v) \equiv_L^f vC_L^f(v, u).$$

If no such words  $u', v'$  exist, we write  $C_L^f(u, v) = C_L^f(v, u) = \infty$ .

Similarly, define the words  $C_R^f(u, v)$  and  $C_R^f(v, u)$  to be the unique elements of  $S^*$  which satisfy  $vu^{-1} \xrightarrow{f}_R C_R^f(u, v)^{-1}C_R^f(v, u)$ , or write  $C_R^f(u, v) = C_R^f(v, u) = \infty$  if no such words exist.

**Definition 5.8** [Dehornoy 1997b, p. 120]. Let  $f : S \times S \rightarrow S^*$  be a complement. We say that  $f$  is *coherent on the left* if, for all  $x, y, z \in S$  such that

$C_L^f(f(x, y), f(x, z)) \neq \infty$  we have

$$C_L^f(f(x, y), f(x, z)) \equiv_L^f C_L^f(f(y, x), f(y, z)).$$

Similarly, we say that  $f$  is *coherent on the right* if, for all  $x, y, z \in S$  such that  $C_R^f(f(z, x), f(y, x)) \neq \infty$  we have

$$C_R^f(f(z, x), f(y, x)) \equiv_R^f C_R^f(f(z, y), f(x, y)).$$

It can be shown [Dehornoy 1997b, Lemma 4] that if an atomic monoid  $M$  can be written  $M = M_L^f$  where the complement  $f$  is coherent on the left, then  $M$  is left cancellative and  $(M, \leq_L)$  is a *quasi-lattice*: every pair of elements  $x, y \in M$  which has a common upper bound ( $z$  such that  $x \leq_L z$  and  $y \leq_L z$ ) has a least upper bound, written  $x \vee_L y$ . This argument is based on Garside's [1969] original argument (see also [Brieskorn and Saito 1972]), and forms the cornerstone of the theory of Garside groups. (The analogous statement when  $M = M_R^g$  is atomic and  $g$  is coherent on the right obviously holds as well.)

We are now ready to state a criterion for a monoid  $M$  to be a Garside monoid:

**Criterion 5.9.** *Let  $M$  be a monoid. Then  $M$  is a Garside monoid if and only if it satisfies the following properties:*

- (C1)  $M$  is finitely generated and atomic.
- (C2) There exist complements  $f : S_1 \times S_1 \rightarrow S_1^*$ , coherent on the left, and  $g : S_2 \times S_2 \rightarrow S_2^*$ , coherent on the right, such that  $M \cong M_L^f$  and  $M \cong M_R^g$ .
- (C3)  $M$  possesses a Garside element, namely an element  $\Delta \in M$  such that the sets  $L(\Delta) = \{x \in M : x \leq_L \Delta\}$  and  $R(\Delta) = \{x \in M : x \leq_R \Delta\}$  are equal and generate  $M$ .

*Proof.* Suppose first that  $M$  satisfies (C1), (C2), and (C3). It follows from [Dehornoy 1997b, Lemma 4] (see the remark above) that  $M$  is left and right cancellative and  $(M, \leq_L)$  is a quasi-lattice. In this situation we may define an operation  $\backslash_L : M \times M \rightarrow M \cup \{\infty\}$  such that  $a(a \backslash_L b) = a \vee_L b$  if  $a$  and  $b$  have a common upper bound, and  $a \backslash_L b = \infty$  otherwise. According to [Dehornoy 2002, Proposition 2.1], the above conditions together with the following condition (D) are sufficient to show that  $M$  is a Garside monoid:

- (D) There exists a finite subset  $P \subset M$  which generates  $M$  and which is closed under the operation  $\backslash_L$  (namely, if  $a, b \in P$  then  $a \backslash_L b \in P$ ).

We show that  $M$  satisfies (D). Let  $P = L(\Delta) = R(\Delta)$ . Note that, by (C3),  $P$  generates  $M$ . Let  $a, b \in P$ . Since  $a \leq_L \Delta$  and  $b \leq_L \Delta$ , we have  $a \vee_L b \leq_L \Delta$ . Let  $c \in M$  such that  $\Delta = (a \vee_L b)c = a(a \backslash_L b)c$ . Then  $(a \backslash_L b)c \leq_R \Delta$ ; thus  $(a \backslash_L b)c \leq_L \Delta$  (since, by (C3),  $L(\Delta) = R(\Delta)$ ); therefore  $(a \backslash_L b) \leq_L \Delta$ , that is  $(a \backslash_L b) \in P$ .

Now suppose that  $M$  is a Garside monoid. Clearly,  $M$  satisfies (C1) and (C3). So, we just need to show that  $M$  satisfies (C2). Choose some finite generating set  $S$  for  $M$ , and consider complements  $f : S \times S \rightarrow S^*$  and  $g : S \times S \rightarrow S^*$  such that

$$\overline{xf(x, y)} = x \vee_L y, \quad \overline{g(x, y)x} = y \vee_R x,$$

for all  $x, y \in S$ . Then, by [Dehornoy and Paris 1999, Theorem 4.1],  $M = M_L^f = M_R^g$ , and, by [Dehornoy 2002, Lemma 5.2],  $f$  is coherent on the left and  $g$  is coherent on the right.  $\square$

It will be convenient, in Section 6, to have the following characterization of a Garside element.

**Lemma 5.10** (Garside elements). *Let  $M$  be a (left and right) cancellative monoid. Then  $\Delta$  is a Garside element (meaning that  $L(\Delta)$  coincides with  $R(\Delta)$  and generates  $M$ ) if and only if the following condition holds:*

(C4)  $L(\Delta) := \{x \in M : x \leq_L \Delta\}$  generates  $M$  and there exists a (necessarily unique) monoid automorphism  $\tau : M \rightarrow M$  such that  $w\Delta = \Delta\tau(w)$  for all  $w \in M$ .

Consequently, we may replace condition (C3) in Criterion 5.9 with condition (C4).

*Proof.* We first show sufficiency. Suppose that (C4) is satisfied. In particular, we have  $\tau(\Delta) = \Delta$  and therefore  $\tau(L(\Delta)) = L(\Delta)$  (since  $\tau$  is a monoid automorphism). On the other hand, by using left and right cancellation one easily obtains from the equation  $x\Delta = \Delta\tau(x)$  that  $\tau(L(\Delta)) = R(\Delta)$ . But then  $L(\Delta) = \tau(L(\Delta)) = R(\Delta)$  and, by hypothesis (C4),  $L(\Delta)$  also generates  $M$ . Thus  $\Delta$  is a Garside element.

Now suppose that  $\Delta$  is a Garside element. By cancellativity and the fact that  $L(\Delta) = R(\Delta)$ , one has a well-defined bijection  $c : L(\Delta) \rightarrow L(\Delta)$  such that  $xc(x) = \Delta$  for all  $x \in L(\Delta)$ . Note that, if  $x \in L(\Delta)$  then so is  $c(x)$  and  $\Delta$  may be written either  $xc(x)$  or  $c(x)c^2(x)$ . Thus  $x\Delta = xc(x)c^2(x) = \Delta c^2(x)$ , for all  $x \in L(\Delta)$ . Since  $L(\Delta)$  generates  $M$ , it follows by cancellativity that the bijection  $c^2$  extends uniquely to a monoid automorphism  $\tau$  satisfying (C4).  $\square$

## 6. Semidirect products

We turn back now to the Artin type representations. Note that, given an Artin representation  $\rho : B_n \rightarrow \text{Aut}(H^{*n})$  associated to a group  $H$  and an element  $h \in H$ , we may form the semidirect product  $H^{*n} \rtimes_{\rho} B_n$ . The aim of this section is to prove the following.

**Theorem 6.1.** *Assume that  $H$  is the group of fractions of a Garside monoid  $M$  and that  $h \in M$  is a Garside element. Let  $G = H^{*n} \rtimes_{\rho} B_n$ , where  $\rho : B_n \rightarrow \text{Aut}(H^{*n})$  denotes the Artin type representation associated to  $(H, h)$  (as defined in the Introduction), and let  $P$  be the submonoid of  $G$  generated by  $M_1 = \phi_1(M)$  and the monoid  $B_n^+$  of positive braids. Then  $P$  is a Garside monoid,  $\Delta = (h_1\sigma_1\sigma_2 \dots \sigma_{n-1})^n$  is a Garside element of  $P$ , and  $G$  is the group of fractions of  $P$ .*

The first step in the proof is finding a presentation for  $H^{*n} \rtimes_{\rho} B_n$ :

**Proposition 6.2.** *Let  $H = \langle S \mid \mathcal{R} \rangle$  be a presentation for  $H$ , and let  $D \in S^*$  be an expression of  $h$ . Then  $G = H^{*n} \rtimes_{\rho} B_n$  has a presentation with generators*

$$S \cup \{\sigma_1, \dots, \sigma_{n-1}\}$$

and relations

$$\begin{aligned} r & \qquad \qquad \qquad \text{for } r \in \mathcal{R}, \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{for } i = 1, \dots, n-2, \\ \sigma_i \sigma_j &= \sigma_j \sigma_i & \text{for } |i-j| \geq 2, \\ \sigma_i x &= x \sigma_i & \text{for } x \in S \text{ and } i = 2, \dots, n-1, \\ x \sigma_1 D \sigma_1 &= \sigma_1 D \sigma_1 D^{-1} x D & \text{for } x \in S. \end{aligned}$$

*Proof of the proposition.* Let  $G_0$  denote the abstract group generated by the union  $S \cup \{\sigma_1, \dots, \sigma_{n-1}\}$ , subject to the relations given in the statement of the proposition. Set  $X = (\bigcup_{i=1}^n \phi_i(S)) \cup \{\sigma_1, \dots, \sigma_{n-1}\}$ . With a little effort one can verify that the mapping  $\varphi : X \rightarrow G_0$  defined by

$$\begin{aligned} \varphi(\phi_i(x)) &= \sigma_{i-1}^{-1} \dots \sigma_1^{-1} D^{i-1} x D^{1-i} \sigma_1 \dots \sigma_{i-1} & \text{for } i = 1, \dots, n \text{ and } x \in S, \\ \varphi(\sigma_i) &= \sigma_i & \text{for } i = 1, \dots, n-1 \end{aligned}$$

determines a homomorphism  $\varphi : G \rightarrow G_0$ , and somewhat more easily that the mapping  $\psi : S \cup \{\sigma_1, \dots, \sigma_{n-1}\} \rightarrow G$  defined by

$$\begin{aligned} \psi(x) &= \phi_1(x) & \text{for } x \in S \\ \psi(\sigma_i) &= \sigma_i & \text{for } i = 1, \dots, n-1 \end{aligned}$$

determines a homomorphism  $\psi : G_0 \rightarrow G$ . One checks without much difficulty that  $(\psi \circ \varphi)(a) = a$  for all  $a \in X$ , and  $(\varphi \circ \psi)(b) = b$  for all  $b \in S \cup \{\sigma_1, \dots, \sigma_{n-1}\}$ ; thus  $\psi \circ \varphi = \text{Id}_G$  and  $\varphi \circ \psi = \text{Id}_{G_0}$ .  $\square$

*Proof of Theorem 6.1.* Let  $\tau : M \rightarrow M$  denote the automorphism of  $M$  induced by conjugation by  $h^{-1}$ , so that  $xh = h\tau(x)$  for all  $x \in M$  (see Lemma 5.10). Let  $S$  be a finite generating set for  $M$ . We may, and do, choose  $S$  so that  $\tau(S) = S$  (for instance we may simply choose  $S$  to be the set of atoms of  $M$ ). Define  $f : S \times S \rightarrow S^*$  so that  $\overline{xf(y)} = \overline{yf(y, x)} = x \vee_L y$  for all pairs  $x, y \in S$ . Similarly define  $g : S \times S \rightarrow S^*$

so that  $\overline{g(x, y)y} = \overline{g(y, x)x} = x \vee_R y$  for all pairs  $x, y \in S$ . As pointed out in the proof of [Criterion 5.9](#), one has  $M = M_L^f = M_R^g$ ,  $f$  is coherent on the left, and  $g$  is coherent on the right. We simply write  $\sim$  for the congruence relation on  $S^*$  defined by the relations in  $M$  (namely,  $\equiv_L^f$ , or equally  $\equiv_R^g$ ). Let  $D \in S^*$  be an expression of  $h$ . Note that for  $x \in S$  we have  $xD \sim D\tau(x)$  and  $\tau^{-1}(x)D \sim Dx$ , where  $\tau(x)$  and  $\tau^{-1}(x)$  also denote elements of the generating set  $S$ . The last family of relations appearing in [Proposition 6.2](#) may be replaced with  $x\sigma_1 D\sigma_1 = \sigma_1 D\sigma_1 \tau(x)$  for all  $x \in S$ , or equivalently with  $\tau^{-1}(x)\sigma_1 D\sigma_1 = \sigma_1 D\sigma_1 x$  for all  $x \in S$ .

Let  $X = S \cup \{\sigma_1, \dots, \sigma_{n-1}\}$ . Let  $\lambda : X \times X \rightarrow X^*$  be the complement defined by

$$\begin{aligned} \lambda(x, y) &= f(x, y) & \text{for } x, y \in S, & & \lambda(\sigma_i, x) &= x & \text{for } x \in S \text{ and } i \geq 2, \\ \lambda(x, \sigma_1) &= \sigma_1 D\sigma_1 & \text{for } x \in S, & & \lambda(\sigma_i, \sigma_j) &= \sigma_j \sigma_i & \text{for } |i - j| = 1, \\ \lambda(\sigma_1, x) &= D\sigma_1 \tau(x) & \text{for } x \in S, & & \lambda(\sigma_i, \sigma_j) &= \sigma_j & \text{for } |i - j| \geq 2, \\ \lambda(x, \sigma_i) &= \sigma_i & \text{for } x \in S \text{ and } i \geq 2, & & & & \end{aligned}$$

and let  $\delta : X \times X \rightarrow X^*$  be the complement defined by

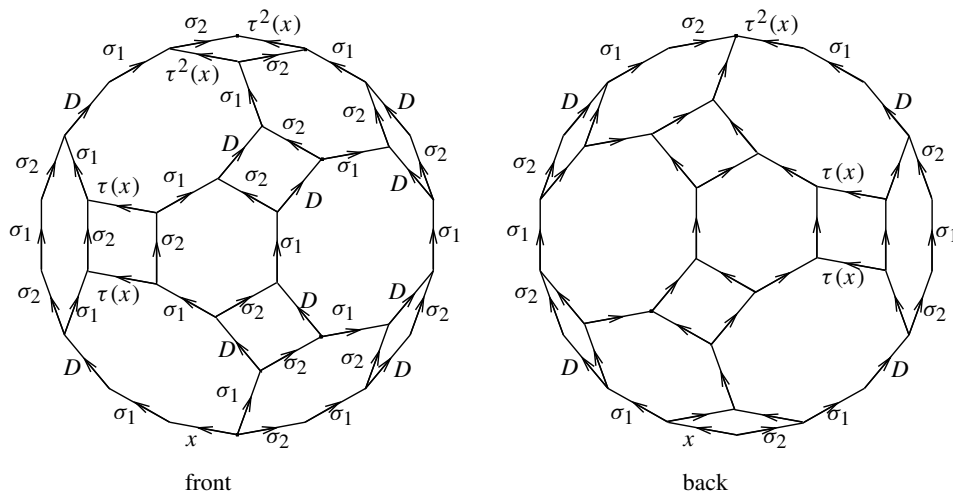
$$\begin{aligned} \delta(x, y) &= g(x, y) & \text{for } x, y \in S, & & \delta(x, \sigma_i) &= x & \text{for } x \in S \text{ and } i \geq 2, \\ \delta(\sigma_1, x) &= \sigma_1 D\sigma_1 & \text{for } x \in S, & & \delta(\sigma_j, \sigma_i) &= \sigma_i \sigma_j & \text{for } |i - j| = 1, \\ \delta(x, \sigma_1) &= \tau^{-1}(x)\sigma_1 D & \text{for } x \in S, & & \delta(\sigma_j, \sigma_i) &= \sigma_j & \text{for } |i - j| \geq 2, \\ \delta(\sigma_i, x) &= \sigma_i & \text{for } x \in S \text{ and } i \geq 2, & & & & \end{aligned}$$

Let  $P_0$  denote the monoid defined by the presentation with generators  $X$  and relations as laid out in [Proposition 6.2](#). Then clearly  $P_0 \cong M_L^\lambda \cong M_R^\delta$ . We denote by  $\approx$  the congruence relation on  $X^*$  defined by the relations of  $P_0$ . (So  $\approx$  is the same congruence relation as  $\equiv_L^\lambda$  and  $\equiv_R^\delta$ ). We now show that  $P_0$  satisfies [Criterion 5.9](#) with complements  $\lambda$  and  $\delta$  and Garside element  $\Delta = (D\sigma_1\sigma_2 \dots \sigma_{n-1})^n$ . It will follow that  $P_0$  is a Garside monoid with group of fractions  $G$  and is canonically isomorphic to the submonoid  $P \subset G$  in the statement of the Theorem.

Clearly  $P_0$  is finitely generated. We check that  $P_0$  is atomic. Let  $\nu : M \rightarrow \mathbb{N}$  be a norm for  $M$ . Let  $\Sigma = \{\sigma_1, \dots, \sigma_{n-1}\}$  and define the function  $\ell : \Sigma^* \rightarrow \mathbb{N}$  by  $\ell(\sigma_{i_1} \dots \sigma_{i_l}) = l$ . We define a function  $\nu_P : X^* \rightarrow \mathbb{N}$  as follows. Let  $w \in X^*$ . Write  $w = u_1 v_1 \dots u_l v_l$ , where  $u_1 \in S^*$ ,  $u_2, \dots, u_l \in S^* \setminus \{\epsilon\}$ ,  $v_1, \dots, v_{l-1} \in \Sigma^* \setminus \{\epsilon\}$ , and  $v_l \in \Sigma^*$ . Then

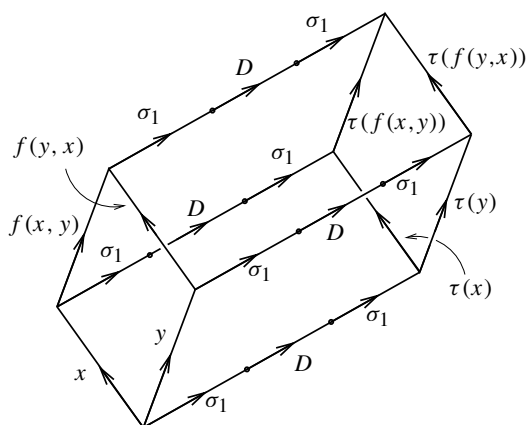
$$\nu_P(w) = \nu(u_1 u_2 \dots u_l) + \ell(v_1 v_2 \dots v_l).$$

One can easily verify that  $\nu_P$  is invariant with respect to all of the relations given in [Proposition 6.2](#), and therefore defines a function  $\nu_P : P_0 \rightarrow \mathbb{N}$ . Moreover, it is easily seen that  $\nu_P$  is a norm, and therefore  $P_0$  is atomic.



**Figure 7.** Left coherence of  $\lambda$  with respect to triple  $\{\sigma_1, \sigma_2, x\}$ . The labels which, for clarity, are missing from the back side may be easily inferred from the relations shown on the front side.

The proof that  $\lambda$  is coherent on the left may be deduced from the existence, for each triple  $\alpha, \beta, \gamma \in X$ , of a certain tiling of the 2-sphere by relations from  $M_L^\lambda$  (i.e., relations of the form  $\alpha\lambda(\alpha, \beta) \approx \beta\lambda(\beta, \alpha)$  for  $\alpha, \beta \in X$ ). We illustrate the two most difficult cases, namely when  $\{\alpha, \beta, \gamma\} = \{\sigma_1, \sigma_2, x\}$  for some  $x \in S$  (Figure 7), and when  $\{\alpha, \beta, \gamma\} = \{\sigma_1, x, y\}$  for some  $x \neq y \in S$  (Figure 8). In the latter case note that, if  $f(x, y)$  is written  $a_1 a_2 \dots a_k$  as a product of generators  $a_i \in S$  then  $\tau(f(x, y)) \approx \tau(a_1)\tau(a_2) \dots \tau(a_k)$  and the face containing  $f(x, y)$  and  $\tau(f(x, y))$  in Figure 8 decomposes into  $k$  faces corresponding to the relations



**Figure 8.** Left coherence of  $\lambda$  with respect to triple  $\{\sigma_1, x, y\}$ .

$a_i \sigma_1 D \sigma_1 \approx \sigma_1 D \sigma_1 \tau(a_i)$ . Similarly for  $f(y, x)$ . The remaining cases are easily handled since in these cases at least one of  $\alpha, \beta, \gamma$  satisfies a commuting relation (explicit in the presentation  $M_L^\lambda$ ) with each of the others.

The proof that  $\delta$  is coherent on the right is similar.

Finally we show that the word  $\Delta = (D\sigma_1\sigma_2\dots\sigma_{n-1})^n$  represents a Garside element of  $P_0$ . We shall employ condition (C4) of Lemma 5.10. Consider the Artin monoid presentation

$$A^+(B_n) = \langle \beta_1, \beta_2, \dots, \beta_n \mid \begin{aligned} &\beta_1\beta_2\beta_1\beta_2 = \beta_2\beta_1\beta_2\beta_1, \\ &\beta_i\beta_{i+1}\beta_i = \beta_{i+1}\beta_i\beta_{i+1} \text{ for } 2 \leq i \leq n-1, \\ &\beta_i\beta_j = \beta_j\beta_i \text{ for } |i-j| \geq 2 \end{aligned} \rangle^+.$$

This monoid  $A^+(B_n)$  is well-known as the Artin monoid of type  $B_n$ , and has Garside element  $\Delta_B = (\beta_1\beta_2\dots\beta_n)^n$ . Clearly there exists a monoid homomorphism  $A^+(B_n) \rightarrow P_0$  such that  $\beta_1 \mapsto D$  and  $\beta_i \mapsto \sigma_{i-1}$  for  $i = 2, 3, \dots, n$ . Thus any relation which is observed in  $A^+(B_n)$  may be deduced in  $P_0$ . In particular, the fact that  $\Delta_B$  is a Garside element in  $A^+(B_n)$  implies that  $\Delta$  is left divisible by  $D, \sigma_1, \dots, \sigma_{n-1}$  and hence is left divisible by every element of  $X$ . It remains to verify that there exists an automorphism  $\tau_P : P_0 \rightarrow P_0$  such that  $w\Delta = \Delta\tau_P(w)$  for all  $w \in P_0$ .

We already know that  $\Delta_B$  is central in  $A^+(B_n)$ . Thus we have  $\sigma_i\Delta = \Delta\sigma_i$  for all  $i = 1, 2, \dots, n-1$ . We may also check (by performing the calculation in  $A^+(B_n)$ ) that

$$\Delta \approx D U^{n-1} \quad \text{where } U := \sigma_1 D \sigma_1 \sigma_2 \sigma_3 \cdots \sigma_{n-1}.$$

Recall that  $\tau$  denotes the automorphism of  $M$  such that, at the level of words,  $xD \sim D\tau(x)$  for all  $x \in S^*$ . Observe also that  $xU \approx U\tau(x)$  for all  $x \in S^*$  (or more loosely speaking, for all  $x \in M$ ). We now define  $\tau_P : P_0 \rightarrow P_0$  such that

$$\begin{aligned} \tau_P(\sigma_i) &= \sigma_i & \text{for } i = 1, 2, \dots, n-1, \\ \tau_P(x) &= \tau^n(x) & \text{for all } x \in M. \end{aligned}$$

It is easily seen that  $\tau_P$  is a monoid isomorphism. Moreover, for all  $x \in M$ ,

$$x\Delta \approx xDU^{n-1} \approx D\tau(x)U^{n-1} \approx DU^{n-1}\tau^n(x) \approx \Delta\tau_P(x),$$

and  $\sigma_i\Delta \approx \Delta\sigma_i$  for all  $i = 1, 2, \dots, n-1$ . Thus condition (C4) of Lemma 5.10 is satisfied, and  $\Delta$  is a Garside element.  $\square$

**Remark.** In closing, we remark that both the above proof and the formulation of Theorem 6.1 were strongly inspired by the example of the Artin group  $A(B_n)$  which, as noted in the introduction, is isomorphic to the semidirect product  $F_n \rtimes B_n$  associated to Artin's 1925 representation [Artin 1925; 1947], namely the Artin type representation associated to  $(\mathbb{Z}, 1)$  (using additive notation). This is evident in both

the description of the fundamental element, and the checking of coherence (see Figures 7 and 8) which follow closely the proof that  $A(B_n)$  has a Garside structure. Note, in particular, that the diagram shown in Figure 7 depicts the Cayley graph for the Coxeter group of type  $B_3$ , once the labels  $D, x, \tau(x)$  and  $\tau^2(x)$  are replaced with a single generator.

In response to a question posed by the referee, we are not aware of any other general constructions of Garside groups obtained in a similar fashion by studying other Artin groups of finite type. We note however that the Artin group of type  $D_n$  is isomorphic to the index 2 torsion free subgroup of the semidirect product  $(C_2)^{*n} \rtimes B_n$  associated to the Artin type representation determined by the nontrivial element of  $C_2$ . However, the group  $C_2$  of order 2 is clearly not Garside (it has torsion!) so that while  $A(D_n)$  admits a Garside structure, this does *not* arise by virtue of Theorem 6.1 just proved. The Artin groups of type  $B_n, n \geq 2$ , would appear to be the only Artin groups of irreducible finite type which are covered in this way by Theorem 6.1.

### Appendix

We denote by  $F_n$  the free group of rank  $n$ , and fix a basis  $x_1, \dots, x_n$  for  $F_n$ .

**Definition.** According to Shpilrain's terminology [2001], a *Wada representation of type (1)* is an Artin type representation associated to  $(\mathbb{Z}, h)$ , where  $h$  is a nonzero integer. Such a representation will be denoted by  $\rho_h^{(1)} : B_n \rightarrow \text{Aut}(F_n)$ . It is determined by

$$\rho_h^{(1)}(\sigma_k)(x_i) = \begin{cases} x_i & \text{if } i \neq k, k+1, \\ x_k^{-h} x_{k+1} x_k^h & \text{if } i = k, \\ x_k & \text{if } i = k+1. \end{cases}$$

The *Wada representation of type (2)* is the representation  $\rho^{(2)} : B_n \rightarrow \text{Aut}(F_n)$  determined by

$$\rho^{(2)}(\sigma_k)(x_i) = \begin{cases} x_i & \text{if } i \neq k, k+1, \\ x_k x_{k+1}^{-1} x_k & \text{if } i = k, \\ x_k & \text{if } i = k+1. \end{cases}$$

and the *Wada representation of type (3)* is the representation  $\rho^{(3)} : B_n \rightarrow \text{Aut}(F_n)$  determined by

$$\rho^{(3)}(\sigma_k)(x_i) = \begin{cases} x_i & \text{if } i \neq k, k+1, \\ x_k^2 x_{k+1} & \text{if } i = k, \\ x_{k+1}^{-1} x_k^{-1} x_{k+1} & \text{if } i = k+1. \end{cases}$$

**Proposition A.3.** (1) Let  $k, l \in \mathbb{Z} \setminus \{0\}$ . Then  $\rho_k^{(1)}$  and  $\rho_l^{(1)}$  are equivalent if and only if  $l = \pm k$ .

(2)  $\rho^{(2)}$  and  $\rho^{(3)}$  are equivalent.

(3) Let  $k \in \mathbb{Z} \setminus \{0\}$ . Then  $\rho^{(2)}$  and  $\rho_k^{(1)}$  are not equivalent.

The following lemmas are preliminary to the proof of this proposition.

**Lemma A.4.** Consider the action of  $B_n$  on  $F_n$  via the representation  $\rho_h^{(1)}$ . For all  $i = 1, \dots, n-1$ , the subgroup of  $F_n$  left fixed by  $\langle \sigma_i \rangle$ , and written  $F_n^{(\sigma_i)}$ , is freely generated by the elements

$$x_1, \dots, x_{i-1}, x_{i+1}^h x_i^h, x_{i+2}, \dots, x_n.$$

removed angle brackets

*Proof.* Write  $F_n = C * D$ , where  $C = \langle x_i, x_{i+1} \rangle$ ,  $D = \langle x_1, \dots, x_{i-1}, x_{i+2}, \dots, x_n \rangle$ . Both groups,  $C$  and  $D$ , are invariant by the action of  $\sigma_i$ . Moreover,  $\sigma_i$  is the identity on  $D$  and acts on  $C$  by  $x_i \mapsto x_i^{-h} x_{i+1}^h x_i^h$ ,  $x_{i+1} \mapsto x_i$ . In particular,  $F_n^{(\sigma_i)} = C^{(\sigma_i)} * D$ .

Let  $u \in C^{(\sigma_i)}$ . Write

$$u = x_i^{n_1} x_{i+1}^{m_1} \dots x_i^{n_r} x_{i+1}^{m_r},$$

where  $r \geq 1$ ,  $m_1, \dots, m_{r-1}, n_2, \dots, n_r \in \mathbb{Z} \setminus \{0\}$ , and  $m_r, n_1 \in \mathbb{Z}$ . First, suppose  $n_1 \neq 0$ . Then

$$\sigma_i(u) = x_i^{-h} x_{i+1}^{n_1} x_i^{m_1} \dots x_{i+1}^{n_r} x_i^{m_r+h} = u;$$

thus

$$-h = n_1, \quad n_1 = m_1, \quad \dots, \quad n_r = m_r, \quad \text{and} \quad m_r + h = 0,$$

hence  $u = (x_{i+1}^h x_i^h)^{-r}$ . Now, suppose  $n_1 = 0$ . Then

$$\sigma_i(u) = x_i^{m_1-h} x_{i+1}^{n_2} x_i^{m_2} \dots x_{i+1}^{n_r} x_i^{m_r+h};$$

thus

$$m_1 - h = 0, \quad m_1 = n_2, \quad n_2 = m_2, \quad \dots, \quad n_r = m_r + h, \quad \text{and} \quad m_r = 0,$$

hence  $u = (x_{i+1}^h x_i^h)^{r-1}$ . □

**Lemma A.5.** Consider the action of  $B_n$  on  $F_n$  via  $\rho_h^{(1)}$ . Then the fixed subgroup  $F_n^{B_n}$  is the cyclic subgroup of  $F_n$  generated by  $x_n^h \dots x_2^h x_1^h$ .

*Proof.* Let  $u \in F_n^{B_n}$ . We have  $u \in F_n^{(\sigma_i)}$  for all  $i = 1, \dots, n-1$ ; thus, by [Lemma A.4](#), the reduced form of  $u$  satisfies the following properties:

- All the exponents are either equal to  $h$  or equal to  $-h$ .
- If  $i \neq 1$ , then  $x_i^h$  is followed by  $x_{i-1}^h$ , and, if  $i \neq n$ , then  $x_i^h$  is preceded by  $x_{i+1}^h$ .
- If  $i \neq n$ , then  $x_i^{-h}$  is followed by  $x_{i+1}^{-h}$ , and, if  $i \neq 1$ , then  $x_i^{-h}$  is preceded by  $x_{i-1}^{-h}$ .

Clearly, these properties hold if and only if  $u$  is of the form  $u = (x_n^h \dots x_2^h x_1^h)^r$ , with  $r \in \mathbb{Z}$ .  $\square$

*Proof of Proposition A.3.* (1) Let  $k \in \mathbb{Z} \setminus \{0\}$ . Let  $\phi : F_n \rightarrow F_n$  be the automorphism determined by  $\phi(x_i) = x_i^{-1}$  for all  $i = 1, \dots, n$ . One can easily verify that

$$\phi^{-1} \circ \rho_k^{(1)}(\sigma_i) \circ \phi = \rho_{-k}^{(1)}(\sigma_i)$$

for all  $i = 1, \dots, n-1$ ; thus  $\rho_k$  and  $\rho_{-k}$  are equivalent.

Let  $k, l > 0$ . For a group  $G$ , we denote by  $H_1(G)$  the abelianization of  $G$ , and, for a subgroup  $H$  of  $G$ , we denote by  $\langle\langle H \rangle\rangle$  the normal subgroup of  $G$  generated by  $H$ . By Lemma A.5, we have

$$F_n / \langle\langle F_n^{\rho_k^{(1)}(B_n)} \rangle\rangle \simeq \langle x_1, \dots, x_n \mid x_n^k \dots x_2^k x_1^k = 1 \rangle;$$

hence

$$H_1(F_n / \langle\langle F_n^{\rho_k^{(1)}(B_n)} \rangle\rangle) \simeq (\mathbb{Z}/k\mathbb{Z}) \times \mathbb{Z}^{n-1}.$$

So, if  $\rho_k^{(1)}$  and  $\rho_l^{(1)}$  are equivalent, then  $(\mathbb{Z}/k\mathbb{Z}) \times \mathbb{Z}^{n-1} \simeq (\mathbb{Z}/l\mathbb{Z}) \times \mathbb{Z}^{n-1}$ ; thus  $k = l$ .

(2) Write

$$y_i = x_1^2 \dots x_{i-1}^2 x_i \quad \text{for } i = 1, \dots, n.$$

One can easily verify that

$$\rho^{(3)}(\sigma_k)(y_i) = \begin{cases} y_i & \text{if } i \neq k, k+1, \\ y_{k+1} & \text{if } i = k, \\ y_{k+1} y_k^{-1} y_{k+1} & \text{if } i = k+1. \end{cases}$$

Let  $\phi : F_n \rightarrow F_n$  be the automorphism determined by  $\phi(x_i) = y_{n-i+1}$  for  $i = 1, \dots, n$ , and let  $\mu : B_n \rightarrow B_n$  be the automorphism determined by  $\mu(\sigma_i) = \sigma_{n-i}$  for  $i = 1, \dots, n-1$ . From the expression of  $\rho^{(3)}(\sigma_k)(y_i)$  given above, follows

$$\phi^{-1} \circ \rho^{(3)}(\sigma_i) \circ \phi = \rho^{(2)}(\mu(\sigma_i))$$

for all  $i = 1, \dots, n-1$ ; thus  $\rho^{(2)}$  and  $\rho^{(3)}$  are equivalent.

(3) Let  $k > 0$ . For  $u \in F_n$ , we denote by  $[u]$  the element of  $H_1(F_n) \simeq \mathbb{Z}^n$  represented by  $u$ . We have

$$\rho^{(2)}(\sigma_1^t)[x_1] = (t+1)[x_1] - t[x_2]$$

for all  $t \in \mathbb{N}$ . On the other hand,  $\rho_k^{(1)}(\beta)$  has finite order as an automorphism of  $H_1(F_n)$ , for all  $\beta \in B_n$ . This shows that  $\rho^{(2)}$  and  $\rho_k^{(1)}$  are not equivalent.  $\square$

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