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AN ALGEBRAIC LOOP THEOREM AND THE DECOMPOSITION OF PD^3-PAIRS

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Abstract

Let (Y, X) denote a 3-dimensional Poincaré pair (PD^3-pair). By the work of Eckmann, Müller and Linnell we may suppose, up to a homotopy equivalence, that the boundary X is a closed 2-manifold. We show that if a component of X fails to be π_1-injective in Y then there is an essential simple loop in X which is nullhomotopic in Y. It follows that there is a finite process of attaching 2-disks along essential simple loops on X, and filling spherical components of X, which transforms (Y, X) into a PD^3-pair (Y', X') with aspherical incompressible boundary X' and such that $\pi_1(Y) = \pi_1(Y')$. The PD^3-pair (X', Y') then admits a canonical decomposition as a connected sum of a finite number of aspherical PD^3-pairs with incompressible boundary, together with a PD^3-pair having virtually free (possibly finite) fundamental group and boundary a (possibly empty) disjoint union of projective planes.

An important breakthrough in the study of 3-manifold topology was made in 1957 when Papakyriakopoulos proved Dehn's Lemma and the Loop and Sphere Theorems [**9**]. The Loop Theorem was subsequently generalised by Stallings [**11**], removing an orientability hypothesis, and a further proof was given by Maskit in [**8**]. In its basic form the Loop Theorem states that, if M is a 3-manifold and B a component of ∂M which is not π_1-injective then there exists a *simple* loop in B which represents a nontrivial element of $\ker(\pi_1(B) \to \pi_1(M))$. In this paper we establish the corresponding statement in the context of 3-dimensional Poincaré pairs. A PD^3 version of the Sphere Theorem is contained in [**3**, **14**]. Together these results enable an approach to the classification of PD^3-pairs via connected sum decomposition.

By a PD^n-complex, or Poincaré complex of formal dimension n, we shall mean a connected finitely dominated[†] CW-complex which exhibits the equivariant Poincaré duality of a closed n-dimensional manifold (with respect, of course to a choice of fundamental class and orientation character). A PD^n-pair, or Poincaré pair, is a finitely dominated connected CW-pair (Y, X) in which X is a disjoint union of PD^{n-1}-complexes and which satisfies the appropriate equivariant Poincaré-Lefschetz duality, again with respect to some chosen fundamental class and orientation character. Guided by the analogy with the known theory of surfaces and 3-manifolds, the study of the homotopy classification and structure of Poincaré complexes and pairs in dimensions 2 and 3 has progressed steadily since these objects were introduced

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[†] As pointed out in [**15**] (c.f. Proposition 1.1), the hypothesis that the complex be finitely dominated is not needed for much of the theory, and amounts to requiring that the fundamental group is finitely presented rather than almost finitely presented. In particular the results from [**3**] which are used here, as well as the proof of Theorem 1 of this paper are independent of this hypothesis.

and first studied by Wall in the 60's. We refer the reader to Wall's recent survey of the subject [**15**] for further details.

It was shown by Eckmann, Müller and Linnell [**4**, **5**] that every Poincaré complex of dimension two is homotopy equivalent to a closed connected surface. On the other hand, there do exist three dimensional Poincaré complexes which are not homotopy equivalent to any closed manifold (see Swan [**12**] and, more recently, Hillman [**6**]). However, the following question remains open: *is every PD^3-pair finitely covered by one which is homotopy equivalent to a compact 3-manifold?*

The connected sum decomposition of PD^3-complexes was considered by the author in [**3**]. On the more general question of decomposing PD^3-pairs, recent progress has been made by Bleile [**1**] in the incompressible boundary case. Extending an earlier result of Turaev [**14**] on PD^3-complexes, Bliele shows that, for a PD^3-pair with incompressible boundary, any splitting of the fundamental group as a nontrivial free product leads to a corresponding connected sum decomposition of the complex (up to homotopy equivalence). This result combined with Theorem 1 (the Algebraic Loop Theorem) leads to a decomposition theorem for PD^3-pairs, Theorem 2, which effectively reduces the "virtual realisation" question mentioned above to the aspherical case: *is every PD^3-group pair (virtually) realised by an aspherical 3-manifold?* (see Section 4 of [**15**] for a discussion of this problem).

We remark that (contrary to what was suggested in [**13**]) the essentially topological arguments, involving cut-and-paste of immersed surfaces in 3-space, which feature at some point in all proofs of the Loop Theorem [**9**, **11**, **8**] cannot be replicated in the context of a PD^3-pair. Nevertheless, it turns out that the classical Loop Theorem may indeed be deduced from a modicum of surface topology, 3-dimensional Poincaré duality, and group theoretic considerations. The key ingredients in the proof of Theorem 1 are:

- A theorem of Maskit [**8**] which classifies the planar covers of a surface, and which translates into Lemma 1 stated below.
- The Kurosh Subgroup Theorem [**7**] which implies that a freely indecomposable subgroup of a free product of groups is either infinite cyclic or conjugate into one of the free factors.
- The Bass-Serre theory of group actions on trees, see [**10**].
- The two principal results of [**3**] which state that (i) each PD^3-complex is homotopy equivalent to a connected sum of aspherical PD^3-complexes and a PD^3-complex with virtually free fundamental group; and (ii) any prime order element of the fundamental group of an orientable PD^3-complex has finite centralizer.

Let (Y, X) denote a PD^3-pair with X a 2-manifold. Let $\widetilde{Y}$ denote the universal covering of Y, and $\widetilde{X}$ the induced covering of X. In [**15**], Lemma 3.3, Wall shows that every component of $\widetilde{X}$ is planar. This, combined with the Theorem of Maskit [**8**] on planar coverings of surfaces shows that the following weak version of the Algebraic Loop Theorem holds.

LEMMA 1 (Weak Loop Theorem). *Let (Y, X) be a PD^3-pair with X a 2-manifold, and let F denote a component of X. Then there exists a finite (possibly empty) family of disjoint orientation preserving simple loops $u_1, \ldots, u_k$ in F, and integers $n_1, \ldots, n_k$, such that the kernel of the map $i_* : \pi_1(F) \to \pi_1(Y)$ (induced by the inclusion $i : F \to Y$) is normally generated by the classes $[u_1]^{n_1}, \ldots, [u_k]^{n_k}$.*

REMARK. In [**2**], Casson and Gordon give a similar statement working modulo the dimension subgroup of $\pi_1(Y)$ over a coefficient field $\mathbf{F}$. Their Theorem 4.6 follows from the above lemma by their observation that a group modulo its dimension subgroup over $\mathbf{F}$ is torsion-free in characteristic 0 and contains only p^r-torsion in characteristic p.

For simplicity, we shall suppose for now that all PD^3-pairs are orientable, and reserve our remarks on the non-orientable case till the end of the paper. Now let F be an oriented closed surface, $u_1, \dots, u_k$ a family of disjoint simple loops in F, and $n_1, \dots, n_k$ positive (nonzero) integers. We consider the group

$$\Gamma \cong \pi_1(F)/ \ll [u_1]^{n_1}, \dots, [u_k]^{n_k} \gg \ .$$

Clearly, we may suppose that the elements in the conjugacy class of $[u_i]$ in Γ have order precisely n_i, for all i, and that no two loops u_i and u_j are parallel.

The group Γ may be viewed as the fundamental group of a 2-complex $\widehat{F}$ obtained from the surface F by attaching a collection of 2-disks $D_1, .., D_k$ where, for $i = 1, .., k$, the 2-disk D_i is attached via a degree n_i map of its boundary onto the loop u_i. By first cutting the surface F along the curves u_i we obtain a number of connected sub-surfaces $F_1, .., F_m$ of F and, for each such subsurface F_r, attaching the disks D_i which are incident on the boundary of F_r yields an orbifold $\widehat{F_r}$. Now $\pi_1(\widehat{F_r})$ is virtually a closed orientable surface group. Moreover, every finite order subgroup is conjugate to one of the cyclic subgroups associated to the boundary curves of F_r. It follows from this discussion that Γ is the fundamental group of a finite graph of groups in which the edge groups are finite cyclic and the vertex groups are virtual closed surface groups.

LEMMA 2. *Let F be an oriented closed surface and, as above, let*

$$\Gamma = \pi_1(F)/ \ll [u_1]^{n_1}, \dots, [u_k]^{n_k} \gg,$$

with $u_1, \dots, u_k$ a nonempty set of mutually disjoint simple loops in F, and $n_1, \dots, n_k$ positive integers. Suppose, in addition, that $[u_i]$ is nontrivial in Γ, for each $i = 1, .., k$. Then

(i) *Γ is freely indecomposable, and*

(ii) *if Γ is virtually free then either*

  (a) *$\Gamma \cong \mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ for some $n \geq 2$ (the case where F is a torus and $k = 1$); or*

  (b) *Γ can be described as the fundamental group of a trivalent graph of groups with vertex groups isomorphic to spherical triangle groups*

$$T(m, n, p) = \langle\, a, b, c \mid a^m = b^n = c^p = 1,\, abc = 1 \,\rangle \text{ with } \frac{1}{m} + \frac{1}{n} + \frac{1}{p} > 1$$

and cyclic edge groups (generated by the standard generators a, b, c in each triangle group), corresponding to a pair of pants decomposition of F along the curves u_i.

Proof. There is clearly no loss of generality in supposing that each simple loop u_i represents a (nontrivial) element of order exactly $n_i \geq 2$ in Γ, and that no two loops u_i, u_j are parallel.

Consider the graph of groups description of Γ given in the preceding discussion. Each edge group is a nontrivial finite cyclic group (generated by $[u_i]$ for some i).

Since the vertex groups are virtual closed surface groups they are freely indecomposable. By application of the Kurosh Subgroup Theorem we see that Γ cannot therefore be a proper free product unless one of its edge groups is trivial. This proves (i).

Now suppose that Γ is virtually free. If some vertex group is infinite then it contains the fundamental group of an orientable surface of nonzero genus as a subgroup. Such groups are not virtually free and so cannot appear as subgroups of Γ. Each vertex group is therefore finite, and so is either a triangle group as described in case (b), or a finite cyclic group arising from a spherical orbifold with 2 singular points. Since there are no parallel curves amongst the u_i, the latter case arises only when F is a torus with a single curve u_1, leading to case (a). Otherwise, all vertices are valence 3 with vertex group and adjacent edge groups as indicated in (b). $\square$

Lemma 3. *If the group Γ satisfies either (a) or (b) of Lemma 2 (ii) then it contains a prime order element with infinite centralizer.*

Proof. We first note that it suffices to find a nontrivial finite subgroup A whose normalizer $N_\Gamma(A)$ in Γ is infinite, for then the centralizer of any prime order element of A contains the centralizer in Γ of A which is finite index in $N_\Gamma(A)$. The lemma is obvious in case (a) where $\Gamma \cong \mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ for $n \geq 2$. We therefore suppose that Γ has a graph of groups structure as in Lemma 2(ii)(b).

Consider a spherical triangle group $T := T(m,n,p) = \langle\, a,b,c \mid a^m = b^n = c^p = 1,\, abc = 1\,\rangle$, and write A, B, C for the cyclic subgroups generated by a, b, c respectively. We have $(m,n,p) = (5,3,2), (4,3,2), (3,3,2)$ or $(m,2,2)$ for $m \geq 2$ (solutions of the inequality $\frac{1}{m} + \frac{1}{n} + \frac{1}{p} > 1$). By inspection, one observes that the subgroup A is index 2 in the normalizer $N_T(A)$ unless $m \leq 3$. Now, suppose that the graph of groups decomposition of Γ has some cyclic edge group A of order $m \geq 4$. Let $T = T(m,n,p)$ and $T' = T(m,n',p')$ denote the vertex groups adjacent to the edge group A. Then Γ contains a subgroup isomorphic to $H = T \star_A T'$, and the normalizer of A in H contains the amalgamated product $N_T(A) \star_A N_{T'}(A)$, which is an infinite group since A is a proper (index 2) subgroup of each factor. Thus the lemma is established in this special case.

We may now suppose that every vertex group in the graph of groups decomposition of Γ is either the alternating group $T(3,3,2) \cong A_4$, the symmetric group $T(3,2,2) \cong S_3$, or the Klein four group $T(2,2,2) \cong C_2 \times C_2$. In the case $T = T(3,3,2)$ the generators a, b of order 3 are conjugate to one another (but generate their own normalizers), while the order 2 subgroup C is a proper (index 2) subgroup of $N_T(C)$. On the other hand, in the case $T = T(3,2,2)$ the order 3 group A is a proper normal subgroup of T. Finally, if $T = T(2,2,2)$ then all cyclic groups are proper normal subgroups.

We view Γ as the group given by a pair of pants (or "pantalon") decomposition of a closed surface F in which each simple loop is labelled either 2, or 3. Cutting along all simple loops carrying the label 2 we obtain connected components as suggested by Figure 1. If all components are as in Figure 1(a) or (b) then all vertex groups are are of type $(3,3,2)$, or $(2,2,2)$. But in this case, repeating the previous argument, we observe that any cyclic edge group of order 2 has infinite centralizer in Γ (since it is a proper subgroup of its normalizer in each adjacent vertex group). Otherwise, there is at least one component which is a chain as shown in Figure 1(c). This

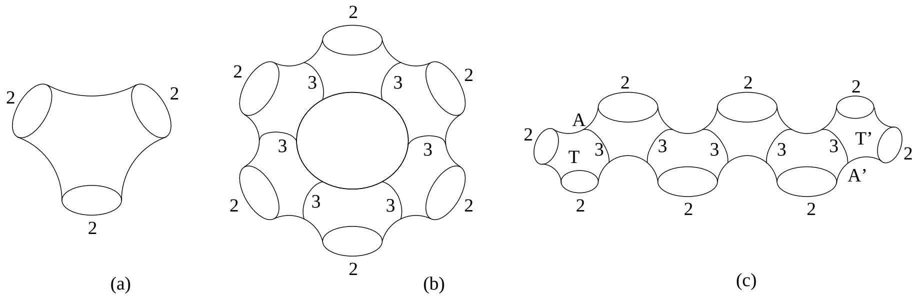

FIGURE 1. *(a) a* $(2,2,2)$*-pantalon; (b) a ring of* $(3,3,2)$*-pantalons; (c) a chain of* $(3,3,2)$*-pantalons, terminated with* $(3,2,2)$*-pantalons.*

chain describes a graph of groups whose fundamental group H embeds (naturally) in Γ. The group H contains distinct vertex groups T and T' each isomorphic to $T(3,2,2)$, and edge groups A and A' (of order 3) such that A is a proper normal subgroup of T, and A' a proper normal subgroup of T'. Moreover, the vertex groups of type $T(3,3,2)$ generate a subgroup B containing both A and A' and H may be written $T \star_A B \star_{A'} T'$. Since the order 3 generators of $T(3,3,2)$ are conjugate to one another, the subgroups A and A' are conjugate in B. Thus $A = bA'b^{-1}$, for some $b \in B$. It follows that the subgroup A is normalised by an order 2 element $\sigma \in T$ as well as by an order 2 element $\widehat{\sigma} \in bT'b^{-1}$. Factoring H by the normal closure of the subgroup B maps the subgroup generated by σ and $\widehat{\sigma}$ homomorphically onto a free product $C_2 \star C_2$. Thus the finite group A has infinite normalizer in Γ. ☐

The following was formulated in [**15**] as “Hope 3.1”.

THEOREM 1 (Algebraic Loop Theorem). *Let* (Y,X) *be an orientable* PD^3*-pair with* X *a 2-manifold, and let* F *denote a component of* X *such that* $i_* : \pi_1(F) \to \pi_1(Y)$ *is not injective. Then there exists a simple loop* u *in* F *which is nullhomotopic in* Y *but not in* F*.*

Proof. For each component F of the boundary X we write Γ_F for the image of $\pi_1(F)$ in $\pi_1(Y)$. Suppose that F is a component which is not π_1-injective. Applying the Weak Loop Theorem, Lemma 1, there exists a (necessarily) non-empty set of disjoint simple loops $u_1,..,u_k$ on F, and positive integers $n_1,..,n_k$, such that

$$\Gamma_F \cong \pi_1(F)/ \ll [u_1]^{n_1}, \ldots, [u_k]^{n_k} \gg \ .$$

Clearly we may suppose that each u_i is homotopically nontrivial in F. We shall show that some u_i is nullhomotopic in Y. Suppose otherwise. Then the hypotheses of Lemma 2 are satisfied for the group Γ_F. We shall use this fact, together with the two main results of [**3**], to derive a contradiction.

Double Y along X to obtain an orientable PD^3-complex $Z = Y \cup_X Y'$. Then $\pi_1(Z)$ may be written as the fundamental group of a graph of groups with a pair of vertices v, v' with $G_v \cong G_{v'} \cong \pi_1(Y)$ and, corresponding to each component X_i of X, an edge e_i from v to v' with edge group $G_{e_i} \cong \Gamma_{X_i}$. The edge-vertex inclusions are given by the inclusion of Γ_{X_i} as a subgroup of $\pi_1(Y)$ in each case. The only fact we need retain from this is that we may view Γ_F is a subgroup of $\pi_1(Z)$.

By [**3**], Corollary 15, the PD^3-complex Z is homotopy equivalent to a connected sum

$$Z \simeq V \# P_1 \# \dots \# P_r$$

where each P_i is an aspherical PD^3-complex and V is a PD^3-complex with virtually free (possibly trivial) fundamental group. Thus $\pi_1(Z)$ is a free product $\pi_1(V) \star G_1 \star \dots \star G_r$, where $G_i \cong \pi_1(P_i)$, for $i = 1, .., r$. In particular, the factors G_i are torsion free.

By Lemma 2, Γ_F is freely indecomposable. It follows from the Kurosh Subgroup Theorem that Γ_F is conjugate (as a subgroup of $\pi_1(Z)$) into one of the factors $\pi_1(V)$, or G_i, or is an infinite cyclic group. However, since it also has nontrivial torsion, it follows that Γ_F is virtually free. By Lemma 2(ii) and Lemma 3, we now have that there exist prime order torsion elements in Γ_F which have infinite centralizer in $\pi_1(Z)$. This contradicts Theorem 17 of [**3**]. ☐

Now suppose that (Y, X) is an oriented PD^3-pair. For the purposes of classification we may suppose that X is a closed 2-manifold (by work of Eckmann, Müller and Linnell [**4**, **5**]) and, furthermore, that X is aspherical. If some component of X is a sphere S^2 then we simply remove the component by filling with a 3-ball. There is no loss of generality in doing this since the PD^3-pair (Y, X) is uniquely determined (up to homotopy equivalence) by the result of such a filling (see Proposition 1.3 of [**15**]). Note that it is for this reason also that the connected sum operation is well defined (see Corollary 1.4 of [**15**]).

Theorem 2. *Let (Y, X) be an orientable PD^3-pair with X an aspherical 2-manifold. Then*

(i) *there exists a finite family of disjoint (essential) simple loops in X such that attaching a 2-disk along each loop yields a PD^3-pair $(\widehat{Y}, \widehat{X})$ with the same fundamental group (each loop is nullhomotopic in Y) and such that each component of $\widehat{X}$ is incompressible (π_1-injective); and*

(ii) *the PD^3-pair $(\widehat{Y}, \widehat{X})$ is homotopy equivalent to the connected sum of a finite number of aspherical PD^3-pairs and a PD^3-complex with virtually free fundamental group.*

Proof. Statement (i) follows from the Algebraic Loop Theorem (Theorem 1 just proved) by a straightforward induction on $|\chi(X)|$. We consider statement (ii).

By the argument of [**3**], Theorem 19, $\pi_1(\widehat{Y})$ is either 1-ended, in which case $(\widehat{Y}, \widehat{X})$ is an aspherical PD^3-pair, or virtually free, in which case $\widehat{X} = \emptyset$ and $\widehat{Y}$ is a PD^3-complex, or $\pi_1(\widehat{Y})$ is a proper free product. In the last of these three cases we may apply a splitting result, stated as Theorem 3.7 in [**15**], which is a consequence of the main result of Bleile [**1**]. It states that if the fundamental group of a PD^3-pair $(\widehat{Y}, \widehat{X})$ with incompressible boundary splits as a free product $G_1 \star G_2$ then $(\widehat{Y}, \widehat{X})$ is homotopy equivalent to the connected sum of PD^3-pairs (Y_1, X_1) and (Y_2, X_2) with $\pi_1(Y_i) \cong G_i$, for $i = 1, 2$. Statement (ii) now follows by a straightforward induction (using the fact that $\pi_1(Y)$ is finitely generated). ☐

We remark that the splitting given by [**15**], Theorem 3.7, is unique up to homotopy. Thus the decomposition in Theorem 2(ii) is unique up to homotopy (and permutation of homotopy equivalent factors).

ADDENDUM (The non-orientable case). We note that Theorem 1 holds equally in the case where (Y, X) is a non-orientable PD^3-pair, and the simple loop u in the Theorem may be supposed orientation preserving. This is a consequence of the fact that the Weak Loop Theorem (Lemma 1) applies in full generality and gives a system of simple loops $u_1, .., u_k$ in the boundary component F which are all orientation preserving. Let (Y', X') denote the orientation double cover of (Y, X), and F' a component of X' covering F. Then each loop u_i lifts to a simple loop in F', and the kernel of the map $\pi_1(F') \to \pi_1(Y')$ is generated by powers of these lifts. The argument given in the proof of Theorem 1 shows that one of these lifts is nullhomotopic in Y', and therefore projects to a simple loop u_i in F which is nullhomotopic in Y.

By the same token Theorem 2 also holds in the non-orientable case, with the single addition that, in part (ii), the connected summand with virtually free fundamental group may possibly be a PD^3-pair with boundary a disjoint union of (incompressible) RP^2's.

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