

Dynamical behavior of Darboux curves

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Abstract

In 1872 G. Darboux defined a family of curves on surfaces of \mathbb{R}^3 which are preserved by the action of the Möbius group and share many properties with geodesics. Here we characterize these curves under the view point of Lorentz geometry and prove some general properties and explicit them on simple surfaces retrieving results of Pell and Santaló.

Introduction

Our interest here is to understand a family of curves on a surface called *Darboux curves*. Almost as geodesics, through every point and direction which is not a principal curvature direction passes a unique Darboux curve.

To understand the dynamic of these curves we will give a natural family of spheres along a Darboux curve in the set of spheres of \mathbb{R}^3 or \mathbf{S}^3 . Then we will study the angle drift of Darboux curves with respect to the foliations of the surface by principal curvature lines and by curves making a constant angle with the principal curvature foliations.

We will also study the Darboux curves on some special surfaces, envelope of very particular one-parameter family of spheres, and on quadrics.

This paper is organized as follows: In section 1 is described the space of spheres of \mathbf{S}^3 and the correspondence with a quadric $\Lambda \subset \mathbb{R}^5$ with the induced Lorentz metric of signature 1. In section 3 is introduced the space of osculating spheres to a surface and the Darboux curves are characterized geometrically. In section 4 a geometric study of Darboux curves on Dupin cyclides is done. In section 5 is obtained the differential equation of Darboux curves in a principal chart and this is applied in subsequent sections. In section 2 a comparative study of quadrics and Dupin cyclides is studied considering a foliation a constant angle with the principal lines. In section 6 is defined a natural plane field in the space of spheres and the study of the integrability is carried out. In section 7 a study of Darboux curves near regular ridge points (zigzag and beak to beak) is done. In 8 is considered the study of Darboux curves in general cylinders, cones and surfaces of revolution viewed as canal surfaces. In section 9 is carried out the global study of Darboux curves on quadrics.

References about these curves are [Da1], [Ri], [Co], [Pe], [En], [Sa1], [Sa2], [Se].

1 The set of spheres in \mathbf{S}^3

The *Lorentz quadratic form* \mathcal{L} on \mathbb{R}^5 and the associated *Lorentz bilinear form* $\mathcal{L}(\cdot, \cdot)$, are defined by $\mathcal{L}(x_0, \dots, x_4) = -x_0^2 + (x_1^2 + \dots + x_4^2)$ and $\mathcal{L}(u, v) = -u_0v_0 + (u_1v_1 + \dots + u_4v_4)$.

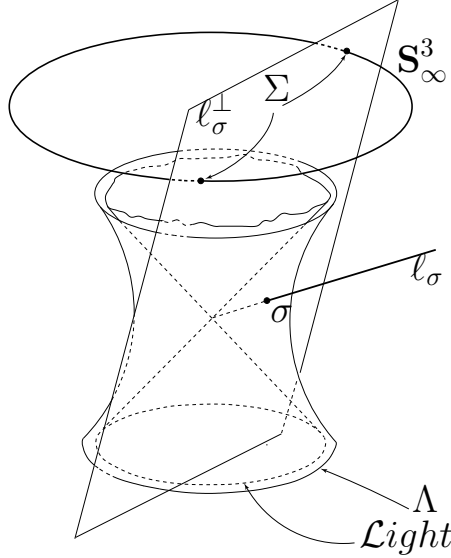


Figure 1: \mathbf{S}_∞^3 and the correspondence between points of Λ and spheres.

The Euclidean space \mathbb{R}^5 equipped with this pseudo-inner product \mathcal{L} is called the *Lorentz space* and denoted by \mathbb{L}^5 .

The isotropy cone $\mathcal{L}i = \{v \in \mathbb{R}^5 \mid \mathcal{L}(v) = 0\}$ of \mathcal{L} is called the *light cone*. Its non-zero vectors are also called *light-like*. The *light cone* divides the set of vectors $v \in \mathbb{L}^5, v \notin \{\mathcal{L} = 0\}$ in two classes:

A vector v in \mathbb{R}^5 is called *space-like* if $\mathcal{L}(v) > 0$ and *time-like* if $\mathcal{L}(v) < 0$.

A straight line is called space-like (or time-like) if it contains a space-like (or respectively, time-like) vector.

The *light cone* $\mathcal{L}ight$ is the cone given by the equation $\mathcal{L} = 0$.

The points at infinity of the light cone in the upper half space $\{x_0 > 0\}$ form a 3-dimensional sphere. Let it be denoted by \mathbf{S}_∞^3 . Since it can be considered as the set of lines through the origin in the light cone, it is identified with the intersection \mathbf{S}_1^3 of the upper half light cone and the hyperplane $\{x_0 = 1\}$, which is given by $\mathbf{S}_1^3 = \{(x_1, \dots, x_4) \mid x_1^2 + \dots + x_4^2 - 1 = 0\}$.

To each point $\sigma \in \Lambda^4$ corresponds a sphere $\Sigma = \sigma^\perp \cap \mathbf{S}_\infty^3$ or $\Sigma = \sigma^\perp \cap \mathbf{S}_1^3$ (see Figure 1). Instead of finding the points of \mathbf{S}^3 “at infinity”, we can also consider the section of the lightcone by a space-like affine hyperplane H_z tangent to the upper sheet of the hyperboloid $\mathcal{H} = \{\mathcal{L} = -1\}$ at a point z . This intersection $\mathcal{L}ight \cap H_z$ inherits from the Lorentz metric a metric of constant curvature 1 (see [H-J], [La-Wa], and Figure 2).

Notice that the intersection of Λ^4 with a space-like plane P containing the origin is a circle $\gamma \subset \Lambda^4$ of radius one in P . The points of this circle correspond to the spheres of a pencil with base circle. The arc-length of a segment contained in γ is equal to the angle between the spheres corresponding to the extremities of the arc.

It is convenient to have a formula giving the point $\sigma \in \Lambda^4$ in terms of the Riemannian geometry of the corresponding sphere $\Sigma \subset \mathbf{S}^3 \subset \mathcal{L}ight$ and a point m on it. For that we need to know also the unit vector \vec{n} tangent to \mathbf{S}^3 normal to Σ at m and the geodesic curvature of Σ , that is the geodesic curvature k_g of any geodesic circle on Σ .

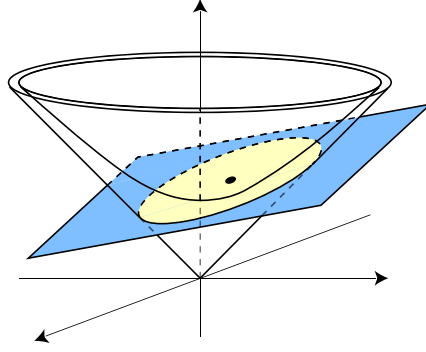


Figure 2: A tangent space to \mathbb{H}^{n+1} cuts the light cone at a unit sphere

Proposition 1. *The point $\sigma \in \Lambda^4$ corresponding to the sphere $\Sigma \subset \mathbf{S}^3 \subset \mathcal{L}ight$ is given by*

$$\sigma = k_g m + \vec{n}. \quad (1)$$

Remark: A similar proposition can be stated for spheres in the Euclidean space \mathbb{E}^3 seen as a section of the light cone by an affine hyperplane parallel to an hyperplane tangent to the light cone.

The proof of Proposition 1 can be found in [H-J] and [La-Oh]. The idea of the proof is shown on Figure 3: Let \mathcal{H} be the affine hyperplane such that $\mathbf{S}^3 = \mathcal{L}ight \cap \mathcal{H}$, let P be the hyperplane such that $\Sigma = \mathbf{S}^3 \cap P$; the vertex of the cone, contained in \mathcal{H} , tangent to \mathbf{S}^3 along Σ is a point of the line P^\perp which contains the point $\sigma \in \Lambda^4$. From Proposition 1, we see that the points in Λ^4 corresponding to a pencil of spheres tangent to a surface M at a point m form two parallel light-rays (one for each choice of normal vector n). Let us now chose the normal vector n , and consider the spheres Σ_k associated to the point $\sigma_k = km + n$.

All the spheres Σ_k , but for at most two, have either a center contact or a saddle contact with M (see Figure 4).

The exceptional spheres correspond to the principal curvatures of M at m , they are called osculating spheres.

When $k \notin [k_1, k_2]$ the intersection of Σ_k and M near the origin reduces to a point, the origin.

When $k \in (k_1, k_2)$ the intersection of Σ_k and M near the origin is formed of two curves intersecting transversely at m .

When $k = k_1$ or k_2 the intersection of Σ_k and M near the origin is formed by a singular curve, in general of cuspidal type, at m .

In fact one can prove the following

Proposition 2. *Let $k = k_1 \cos^2 \alpha + k_2 \sin^2 \alpha$. Then the angle of the tangents at m to $\Sigma_k \cap M$ with the principal direction corresponding to k_1 is $\pm \alpha$.*

In general, the intersection of an osculating sphere with M admits a cuspidal point at m , the tangent to the cusp is then the principal direction associated to the curvature k_i . In any case, when $k \in [k_1, k_2]$ goes to k_i , the two tangent directions at m to $\Sigma_k \cap M$ converge to the principal direction associated to the curvature k_i . From that we can see that lines of curvature as osculating spheres are conformally defined.

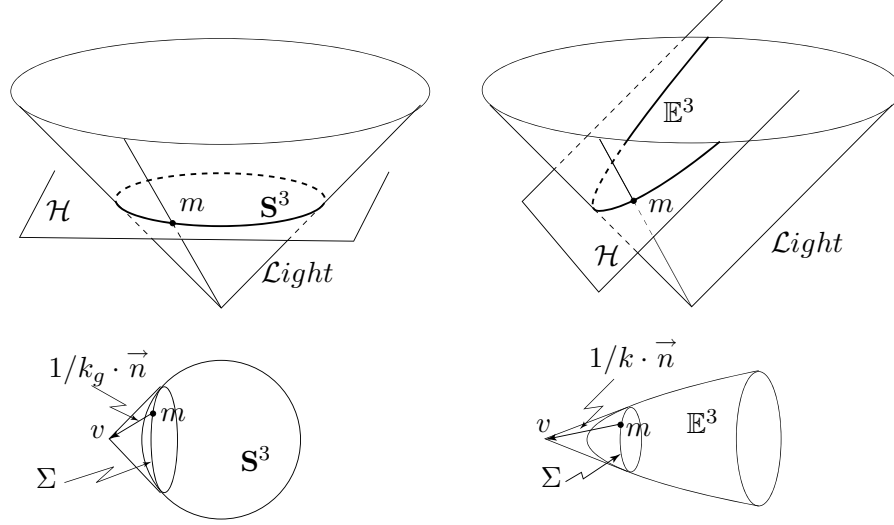


Figure 3: Spherical and Euclidean models in the Minkowski space \mathbb{L}^5 (up). The geodesic curvature k_g , picture in the affine hyperplane \mathcal{H} (down).

2 Foliations making a constant angle with respect to Principal Foliations

Consider a surface M with principal foliations \mathcal{P}_1 and \mathcal{P}_2 and umbilic set \mathcal{U} , and the foliation \mathcal{F}_α^+ and \mathcal{F}_α^- introduced in section 4. Recall that the leaves of \mathcal{F}_α make angle α and $-\alpha$ with \mathcal{P}_1 . So will write $\mathcal{F}_\alpha = \{\mathcal{F}_\alpha^+, \mathcal{F}_\alpha^-\}$.

2.1 Foliations \mathcal{F}_α in Dupin cyclides

Dupin cyclides are very special: they are surfaces which are in two different ways envelopes of one-parameter families of spheres (see [Da3]). This implies that the corresponding curves are circles or hyperbolas in Λ^4 , intersection of Λ^4 with an affine plane (see [La-Wa]).

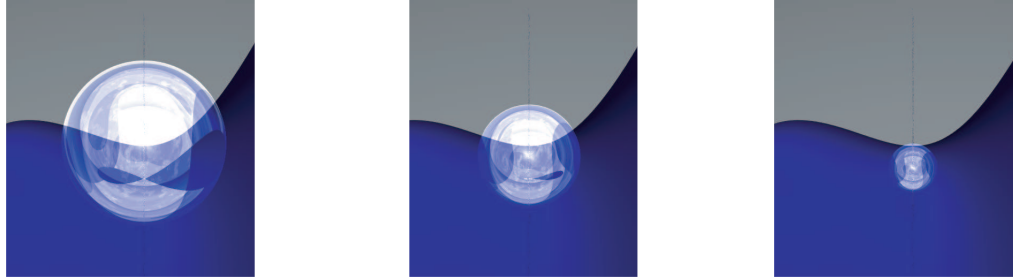
There are three types of Dupin cyclides. One can chose a nice representant of each class:

- A) The boundary of a tubular neighbourhood of a geodesic of \mathbf{S}^3
- B) A cylinder of revolution in \mathbb{R}^3
- C) A cone of revolution in \mathbb{R}^3 .

Then, in cases A) and B) the foliations \mathcal{F}_α^+ and \mathcal{F}_α^- are totally geodesic foliations.

In the case B), four foliations are foliation by circles: the two foliation by characteristic circles \mathcal{F}_0 and $\mathcal{F}_{\pi/2}$, and two others: the foliations by Villarceau circles.

In the case C), we have to develop the cone on a plane –this is a local isometry out of the origin– to see that a foliation of the plane by curves making a contant angle with rays is a foliation by logarithmic spirals. The picture on the cone is obtained by rolling the plane foliation back on the cone.

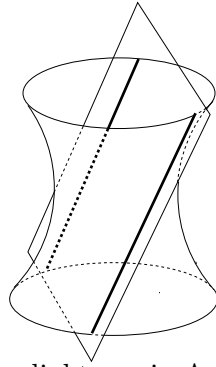


Saddle contact

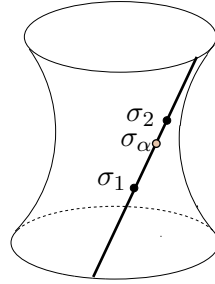
Contact with an osculating sphere

Center contact

Figure 4: Possible contacts of a sphere and a surface



lightrays in Λ
corresponding to a
pencil of tangent spheres



lightray in Λ
corresponding to
oriented spheres tangent to M at m

Figure 5: Tangent spheres

2.2 Foliations \mathcal{F}_α in quadrics

Proposition 3. *Consider an ellipsoid $\mathbb{E}_{a,b,c}$ with three axes $a > b > c > 0$. Then $\mathbb{E}_{a,b,c}$ have four umbilic points located in the plane of symmetry orthogonal to middle axis; they are of the type D_1 , i.e., a singularity of index $1/2$ and having one separatrix for the principal curvature lines. For all α the singularities of \mathcal{F}_α are the four umbilic points and this configuration is topologically equivalent to the principal configuration \mathcal{P} of the ellipsoid near the umbilic points which are D_1*

Proof. Without lost of generality suppose that $\mathbb{E}_{a,b,c}$ is defined by the equation $H(x, y, z) = \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1$, with $a > b > c > 0$.

Consider the parametrization of the ellipsoid in a neighborhood of the umbilic point $p_0 = (u_0, 0, v_0) = (\sqrt{\frac{a(a-b)}{a-c}}, 0, \sqrt{\frac{c(c-b)}{c-a}})$.

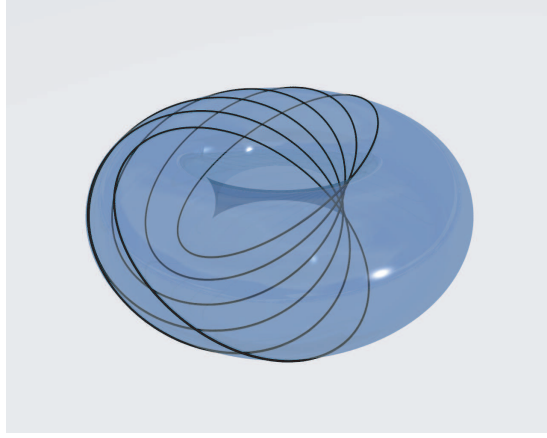


Figure 6: Foliation of a torus of revolution by Villarceau circles

$$\alpha(u, v) = p_0 + uE_1 + vE_2 + \frac{1}{2}[\sqrt{ac}(u^2 + v^2) + \frac{\sqrt{c(b-c)(a-b)}}{b^3}(u^3 + uv^2) + h.o.t]E_3 + h.o.t.$$

Here $\{E_1, E_2, E_3\}$, $E_2 = (0, 1, 0)$, is a positive orthonormal base and the ellipsoid is oriented by $E_3 = -\nabla H(p_0)/|\nabla H(p_0)$.

In a neighborhood of the umbilic point $(0, 0)$ the differential equation of the foliation \mathcal{F}_α in the chart (u, v) is given by:

$$A(u, v)dv^2 + B(u, v)dudv + C(u, v)du^2 = 0, \text{ where}$$

$$A(u, v) = -u - \cos 2\alpha \sqrt{u^2 + v^2} + R_3(u, v) + A_2(u, v)$$

$$B(u, v) = 2v + B_2(u, v)$$

$$C(u, v) = u - \cos 2\alpha \sqrt{u^2 + v^2} + R_3(u, v) + C_2(u, v)$$

Here $A_2 = O(r^2)$, $C_2 = O(r^2)$ and $R_3(u, v) = O(r^3)$, $r = \sqrt{u^2 + v^2}$.

The above implicit equation has, real separatrices with limit direction given by $\pm 2\alpha$ and the behavior of the integral curves near 0 is the same of an umbilic point of type D_1 .

In fact, consider the blowing-up $u = r \cos \alpha$, $v = r \sin \alpha$.

The differential equation of \mathcal{F}_α in the new variables is given by:

$$(\cos 2\alpha - \cos \alpha + rR_1(r, \alpha))dr^2 + r(2 \sin \alpha + rR_2(r, \alpha))drd\alpha + r^2(\cos 2\alpha + \cos \alpha + rR_3(r, \alpha))d\alpha^2 = 0.$$

The two singular points are given by $(2\alpha, 0)$ and $(-2\alpha, 0)$. Direct analysis shows that both singular points are hyperbolic saddles of the adapted vector fields to the implicit equation near these singularities. The blowing-down of the saddle separatrices are the umbilic separatrices of \mathcal{F}_α^+ and \mathcal{F}_α^- . See Fig. 7.

Therefore it follows that the pair of foliations \mathcal{F}_α^+ and \mathcal{F}_α^- near an umbilic point of the ellipsoid with three distinct axes is topologically equivalent to the configuration of principal lines near a Darbouxian umbilic point D_1 . \square

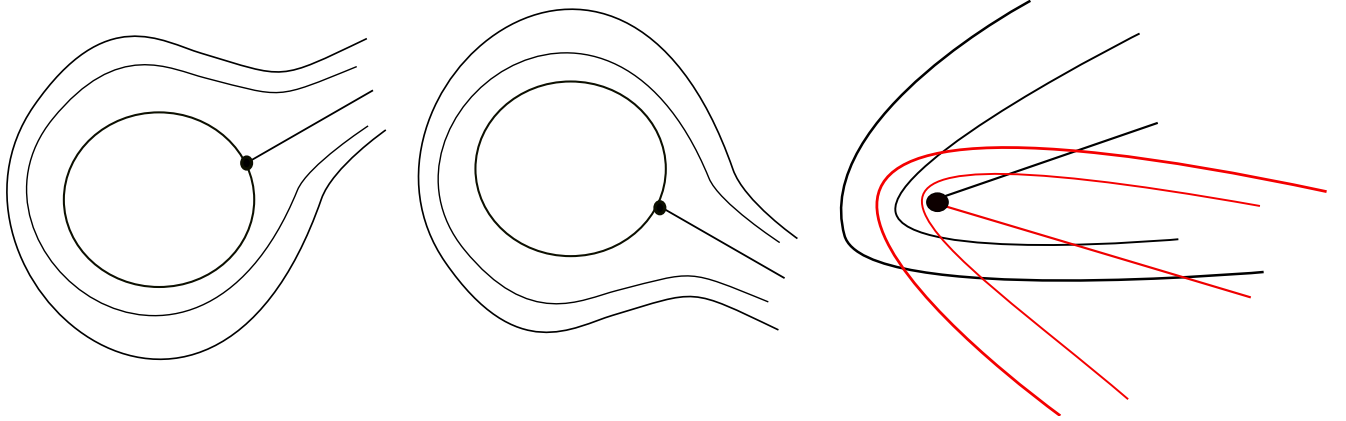


Figure 7: Resolution of the foliations \mathcal{F}_α^+ \mathcal{F}_α^- .

Remark: The study of principal lines was first considered by Monge, see [Mo1] and [Mo2]. Near umbilic points the behavior of principal lines on real analytic surfaces was established by Darboux, [Da2]. See also [Gu], [G-S], [Ga-S3] and references therein.

Proposition 4. Consider an ellipsoid $\mathbb{E}_{a,b,c}$ with three axes $a > b > c > 0$. On the ellipse $\Sigma_{xz} \subset \mathbb{E}_{a,b,c}$, containing the four umbilic points, p_i , ($i = 1, \dots, 4$) counterclockwise oriented, denote by $s_1(\alpha) = 2 \int_c^b \sin \alpha \left[\frac{\sqrt{u}}{-H(u)} \right] du$ (resp. $s_2(\alpha) = 2 \int_b^a \cos \alpha \left[\frac{\sqrt{v}}{H(v)} \right] dv$) a distance between the adjacent umbilic points p_1 and p_4 (resp. p_1 and p_2). Define $\rho(\alpha) = \frac{s_2(\alpha)}{s_1(\alpha)}$.

Then if $\rho \in \mathbb{R} \setminus \mathbb{Q}$ (resp. $\rho \in \mathbb{Q}$) all the leaves of \mathcal{F}_α are recurrent (resp. all, with the exception of the umbilic separatrices, are closed). See Fig. 8.

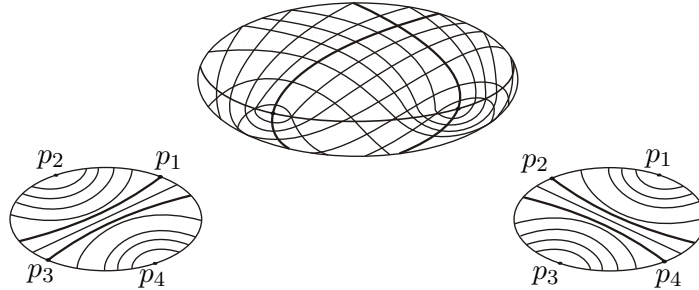


Figure 8: Foliations \mathcal{R}_i of the ellipsoid $\mathbb{E}_{a,b,c}$

Proof. The ellipsoid $\mathbb{E}_{a,b,c}$ belongs to the triple orthogonal system of surfaces defined by the one parameter family of quadrics, $\frac{x^2}{a-\lambda} + \frac{y^2}{b-\lambda} + \frac{z^2}{c-\lambda} = 1$ with $a > b > c > 0$, see also [Sp] and [St]. The following parametrization $\alpha(u, v) = (x(u, v), y(u, v), z(u, v))$ of $\mathbb{E}_{a,b,c}$, where

$$x(u, v) = \pm \sqrt{\frac{a(u-a)(v-a)}{(b-a)(c-a)}}, \quad y(u, v) = \pm \sqrt{\frac{b(u-b)(v-b)}{(b-a)(b-c)}}, \quad z(u, v) = \pm \sqrt{\frac{c(u-c)(v-c)}{(c-a)(c-b)}}, \quad (2)$$

defines the principal coordinates (u, v) on $\mathbb{E}_{a,b,c}$, with $u \in (b, a)$ and $v \in (c, b)$.

The first fundamental form of $\mathbb{E}_{a,b,c}$ is given by:

$$I = ds^2 = Edu^2 + Gdv^2 = \frac{(v-u)u}{4H(u)}du^2 + \frac{(u-v)v}{4H(v)}dv^2 \quad (3)$$

The second fundamental form with respect to the normal $N = -(\alpha_u \wedge \alpha_v)/|\alpha_u \wedge \alpha_v|$ is given by

$$II = edu^2 + gdv^2 = \frac{(v-u)}{4H(u)}\sqrt{\frac{abc}{uv}}du^2 + \frac{(u-v)}{4H(v)}\sqrt{\frac{abc}{uv}}dv^2 \quad (4)$$

$$H(x) = (x-a)(x-b)(x-c)$$

Therefore the principal curvatures are given by:

$$k_1 = \frac{e}{E} = \frac{1}{u}\sqrt{\frac{abc}{uv}}, \quad k_2 = \frac{g}{G} = \frac{1}{v}\sqrt{\frac{abc}{uv}}.$$

The four umbilic points are given by: $(\pm x_0, 0, \pm z_0) = (\pm\sqrt{\frac{a(a-b)}{a-c}}, 0, \pm\sqrt{\frac{c(c-b)}{c-a}})$.

The differential equation of the foliation \mathcal{F}_α in the principal chart (u, v) is given by

$$H(u)v \cos^2 \alpha dv^2 + H(v)u \sin^2 \alpha du^2 = 0 \Leftrightarrow \frac{v}{H(v)} \cos^2 \alpha dv^2 + \sin^2 \alpha \frac{u}{H(u)} du^2 = 0.$$

Define $d\sigma_1 = \sin \alpha \sqrt{\frac{u}{(-H(u))}} du$ and $d\sigma_2 = \cos \alpha \sqrt{\frac{v}{H(v)}} dv$

Therefore the differential equation of \mathcal{F}_α is equivalent to $d\sigma_2^2 - d\sigma_1^2 = 0$ in the rectangle $[0, s_1(\alpha)] \times [0, s_2(\alpha)]$.

On the ellipse $\Sigma = \{(x, 0, z) | \frac{x^2}{a} + \frac{z^2}{c} = 1\}$ define a distance between the umbilic points $p_1 = (x_0, 0, z_0)$ and $p_4 = (x_0, 0, -z_0)$ by $s_1(\alpha) = 2 \int_c^b \sin \alpha [\frac{\sqrt{u}}{(-H(u))}] du$ and that between the umbilic points $p_1 = (x_0, 0, z_0)$ and $p_2 = (-x_0, 0, z_0)$ is defined by $s_2(\alpha) = 2 \int_b^a [\cos \alpha \frac{\sqrt{v}}{H(v)}] dv$.

The ellipse Σ is the union of four umbilic points and four principal umbilical separatrices for the principal foliations. So $\Sigma \setminus \{p_1, p_2, p_3, p_4\}$ is a transversal section of the foliation \mathcal{F}_α .

Therefore near the umbilic point p_1 the foliation, say \mathcal{F}_α^1 , with umbilic separatrix contained in the region $\{y > 0\}$ define a the return map $\sigma_+ : \Sigma \rightarrow \Sigma$ which is an isometry, reverting the orientation, with $\sigma_+(p_1) = p_1$. This follows because in the principal chart (u, v) this return map is defined by $\sigma_+ : \{u = b\} \rightarrow \{v = b\}$ which satisfies the differential equation $\frac{ds_2}{ds_1} = -1$. By analytic continuation it results that σ_+ is a isometry reverting orientation with two fixed points $\{p_1, p_3\}$. The geometric reflection σ_- , defined in the region $y < 0$ have the two umbilic $\{p_2, p_4\}$ as fixed points. So the Poincaré return map $\pi_1 : \Sigma \rightarrow \Sigma$ (composition of two isometries σ_+ and σ_-) is a rotation with rotation number given by $s_2(\alpha)/s_1(\alpha)$.

Analogously for \mathcal{F}_α^2 with the Poincaré return map given by $\pi_2 = \tau_+ \circ \tau_-$ where τ_+ and τ_- are two isometries having respectively $\{p_2, p_4\}$ and $\{p_1, p_3\}$ as fixed points. \square

Remark: The special case $\alpha = \pi/4$ was studied in [Ga-S1]. A more general framework of implicit differential equations, unifying various families of geometric curves was studied in [Ga-S2]. See also [Ga-S3].

Proposition 5. *In any surface, free of umbilic points, the leaves of \mathcal{F}_α^+ and \mathcal{F}_α^- are Darboux curves if and only if the surface is conformal to a Dupin cyclide.*

Proof. From the differential equation of Darboux curves it follows that:

$$3(k_1 - k_2) \sin \alpha \cos \alpha \frac{d\alpha}{ds} = \frac{1}{\sqrt{E}} \frac{\partial k_1}{\partial u} \cos^3 \alpha + \frac{1}{\sqrt{G}} \frac{\partial k_2}{\partial v} \sin^3 \alpha.$$

So all Darboux curves are leaves of \mathcal{F}_α^\pm if and only if $\frac{\partial k_1}{\partial u}(u, v) = \frac{\partial k_2}{\partial v}(u, v) = 0$. By Proposition 24 this is exactly the condition that characterizes the Dupin cyclides. \square

3 Spheres tangent to a surface along a curve

Above each point $m \in M$ which is not an umbilic, the spheres tangent to the surface M having a saddle contact with M form an interval of boundary the two osculating spheres at m . Let us first define the 3-dimensional subset $V(M) \subset \Lambda^4$ as the union of the spheres having a saddle contact with the surface M and the osculating spheres.

Therefore $V(M)$ is an interval fiber-bundle $\pi V(M) \rightarrow M$ over M ; the boundary of $V(M)$ is the surface \mathcal{O} of spheres osculating M (this surface has two connected components when the surface M has no umbilical point). $V(M)$ is a 3-manifold with boundary a surface \mathcal{O} made of spheres osculating M (this surface has two connected components when the surface M has no umbilical point). $V(M)$ inherits from Λ^4 a semi-Riemannian metric. At each point, the kernel direction is the direction of the light ray through the point.

Let us consider now a curve $C \subset M^2$.

The restriction of $V(M)$ to C form a two-dimensional surface in Λ^4 which is a light-ray interval bundle $V(C)$ over C out of the umbilical points of M which may belong to C . From $V(M)$, we get on $V(C)$ an induced semi-Riemannian metric.

In this text we will use $'$ for derivatives with respect to parametrization of curves contained in \mathbb{R}^3 or \mathbf{S}^3 , (often the parameter is an arc length), and \cdot for derivatives with respect to a parametrization of a curve in Λ^4 (often the parameter is an arc length, but now for the metric induced from the Lorentz "metric").

One section of this bundle has the property that at every point m where the curve is not tangent to a principal direction, the point $\sigma_{C'}(m)$ correspond to a sphere $\Sigma(m)$ such that one branch of the intersection $\Sigma_{C'}(m) \cap M$ is tangent to C at m . If at m the curve C is tangent to a principal direction we take the corresponding osculating sphere as $\Sigma_{C'}(m)$.

We call this section $Cansec(C)$ of $V(C)$ the canonical section, and the corresponding family of spheres the canonical family along C ; the envelope $CanCan(C)$ of the spheres of this family is called canonical canal corresponding to $C \subset M$.

Proposition 6. *Consider the notation above. Then the following holds:*

- i) The section $Cansec(C)$ satisfies $\mathcal{L}(\vec{k}_g) > 0$, and moreover $\vec{k}_g \in T_m V(M)$.*
- ii) The section $Cansec(C)$ is a geodesic in $V(C)$. It is of minimal length among sections of $V(C)$ therefore it. Moreover it is the unique section on $V(C)$ which is of critical length.*
- iii) One of the cuspidal edges of the envelope $CanCan(C)$ is C .*

Proof. The condition defining the sphere $\Sigma_{C'}(m)$ implies that the contact order of C and $\Sigma_{C'}(m)$ is at least 2, one more that the contact order of $\Sigma_{C'}(m) \cap M$ and C which is at least one. To

see that, notice that, in terms of the arc-length of a branch of $\Sigma_{C'}(m) \cap M$, or equivalently of the arc-length s on C , the angle of $\Sigma(m)$ with M along $\Sigma_{C'}(m) \cap M$ is of the order of s , if not smaller. The distance to C is of the order of s^2 , if not smaller. Therefore the order of the distance of a point of C to $\Sigma_{C'}(m)$ is of order s^3 if not smaller. This means that C and the sphere $\Sigma_{C'}(m)$ have contact of order at least 2 at m .

Definition 7. *The curves on M such that at each point the osculating sphere is tangent to M will be called Darboux curves.*

Recall a lemma from [L-S]

Lemma 8. *A curve $\Gamma(t) = \text{span}(\gamma(t))$ has contact of order $\geq k$ with a sphere Σ corresponding to σ iff*

$$\sigma \perp \text{span}(\gamma(t), \dot{\gamma}(t), \dots, \gamma^{(k)}(t))$$

Proof. The sphere Σ is the zero level of the function $f(x) = \langle x | \sigma \rangle$. Then the contact of $\Gamma(t)$ and Σ has the order of the zero of $(f \circ \gamma)(t) = \langle \gamma(t) | \sigma \rangle$. \square

Therefore, the sphere $\text{Cansec}(m)$ is orthogonal to m , \dot{m} and \ddot{m} . \square

Differentiating the relation $\langle \sigma | \dot{m} \rangle = 0$ and using the relation $\langle \sigma | \ddot{m} \rangle = 0$ we get $\langle \dot{\sigma} | \dot{m} \rangle = 0$. Differentiating the relation $\langle \dot{\sigma} | m \rangle = 0$, and using the relation $\langle \dot{\sigma} | \dot{m} \rangle = 0$, we get the relation $\langle \ddot{\sigma} | m \rangle = 0$.

We can use as parameter the arc-length of the space-like curve $\sigma = \text{Cansec}$. Its geodesic curvature vector, orthogonal to \hat{m} is therefore orthogonal to $\hat{m} \oplus \mathbb{R}\dot{\sigma}$, proving that $\{\sigma\}$ is a geodesic in $V(C)$.

We notice that $\ddot{\sigma}$ is tangent to $T_m V(M)$, as $T_m(V(M)) = T_m(\mathcal{L}ight) = \hat{m}^\perp$.

Now we will use the Darboux frame T, N_1, n, m of the curve $C \subset M \subset \mathbf{S}^3 \subset \mathcal{L}ight$, where N_1 is the unit vector tangent to M normal to C compatible with the orientation of M . Using again the formula $\sigma = k_n m + n$, where k_n is the normal curvature in the direction tangent to C at m , we see that, when σ is the section $\text{Cansec}(C)$

$$|\sigma'| = |k'_n m + k_n m' + n'| = |k'_n m - \tau_g N_1| = |\tau_g| \quad (5)$$

where the *geodesic torsion* τ_g is defined by the following formula

$$((\nabla_T n) | N_1) = -\tau_g. \quad (6)$$

Observe that our formula (5) gives an interpretation of the geodesic torsion of a curve $C \subset M$ as the rotation speed of the canonical family of spheres tangent to M along C .

In order to compute the geodesic curvature vector of the section $\text{Cansec}(C)$, we need to use its parametrization by arc-length in Λ^4 . Then $\dot{\sigma} = \sigma' \frac{1}{\tau_g} = \frac{1}{\tau_g} (k'_n m - \tau_g N_1)$. Differentiating once more, we get

$$\ddot{\sigma} = -\frac{\dot{\tau}_g}{\tau_g^3} (k'_n m - \tau_g N_1) + \frac{1}{\tau_g^2} [k''_n m + k'_n T - \tau'_g N_1 - \tau_g (-k_g T + \tau_g n)],$$

notice that k_g is the geodesic curvature of $C \subset M$, while $\overrightarrow{k_g}$ is the geodesic curvature vector of the curve $\text{Cansec} \subset \Lambda^4$.

Simplifying, we get

$$\ddot{\sigma} = \phi(s)m + \frac{1}{\tau_g^2}[(k'_n + \tau_g k_g)T - \tau_g^2 n]$$

As $\vec{k}_g = \ddot{\sigma} + \sigma$ we get

$$\vec{k}_g = \psi(s)m + \frac{1}{\tau_g^2}(k'_n + \tau_g k_g)T \quad (7)$$

As the vector \vec{k}_g is orthogonal to m and to $\dot{\sigma} = \frac{1}{\tau_g}(k'_n m - \tau_g N_1)$, we see that the curve $Cansec$ is a geodesic on $V(C)$.

Otherwise, when $\sigma = km + n$, we see that spheres tangent a surface along a curve form a space-like curve in Λ^4 ; explicitly we get

$$|\dot{\sigma}| = \left| \frac{1}{\tau_g}[(k - k'_n)m + (k - k_n)T + \tau_g N_1] \right|, \quad (8)$$

giving, as m, T and N are mutually orthogonal, a proof of the fact the section $Cansec(C)$ has minimal length among sections, as the three vectors m, T, N are mutually orthogonal in \mathbb{L}^5 . Formula 8 shows also that no other section of $V(C)$ is of critical length.

Remark: The characteristic circle of the envelope is the intersection of $span(\sigma, \dot{\sigma})^\perp$ and the sphere $\mathbf{S}^3 \subset \mathbb{R}^4$; as the vector T is orthogonal to m, n and N_1 , and therefore to σ and σ' (and also $\dot{\sigma}$), the characteristic circle is tangent at m to C .

Recall that a *drill* ([Tho], [L-S]) is a curve in the space of spheres the geodesic curvature of which is light-like at each point. Generically, points of a drill are osculating spheres to the curve $C \in \mathbf{S}^3$ defined by the geodesic acceleration vector \vec{k}_g of the drill.

We see that if we can find drills in $V(m)$ we find geodesics. In fact we find that way almost all of them.

Formula 7 implies, as m is the normal direction to $V(M)$ at a point $\sigma = km + n$, the following theorem.

Theorem 9. *The curve C is a Darboux curve if and only if the section $Cansec(C) \subset V(M)$ is a geodesic in $V(M)$. This happens if and only if the curve $C \subset M$ satisfies the equation:*

$$k'_n + \tau_g k_g = 0. \quad (9)$$

Remark: The light ray of $V(M)$ are also geodesic. Segments of geodesics of $V(M)$ which are not tangent to light rays define an arc of curve $C \subset M$, and therefore, when \vec{k}_g is light-like, C is a piece of Darboux curve on M .

Also when C is a Darboux curve the only cuspidal edge of the envelope $CanCan(C)$ is C . See [Tho].

Definition 10. *We denote by $\Sigma_{m,\ell}$ the sphere tangent to the surface M at the point m such than one branch of the intersection $\Sigma_{m,\ell} \cap M$ is tangent to the direction ℓ (canonical sphere associated to the direction ℓ). We will also use the notation $\Sigma_{m,v}$ when the direction ℓ is generated by a non-zero vector v .*

Given a curve $C \subset M$ such that the tangent vector to C at $c(t)$ is contained in ℓ , $\Sigma_{c(t), \dot{c}(t)}$ is, among the spheres tangent to M at m , the one which have the best contact at m with the curve C .

Remark: $\tau_g ds$ is the differential of the rotation of the sphere $\Sigma_{c(t), \dot{c}(t)}$ along the curve C .

Proof: The sphere $\Sigma_{c(t), \dot{c}(t)}$ has a tangent movement which is a rotation of “axis” the characteristic circle of the family, which is tangent to C . It is tangent to the tangent plane. We see that changing the sphere tangent to the surface along C changes the “pitch” term but not the “roll” term. \square

Proposition 11. *Let C be a curve contained in the surface $M \subset \mathbf{S}^3$. Let γ_C be the curve in Λ^4 obtained considering at each point $m \in C$ the sphere $\Sigma_{m, \ell(m)}$, where $\ell(m)$ is the direction tangent at m to C . Suppose that the curve C is nowhere tangent to a principal direction of curvature. Then the curve γ_C is space-like with at every point a space-like geodesic acceleration. Moreover the curve C is one fold of the singular locus of the canal surface defined by γ_C .*

Proof: The condition defining the canonical sphere $\Sigma_{c(t), \dot{c}(t)}$ guarantees that the characteristic circle $CC(t)$ is tangent to C , which is therefore the singular curve of the canal. \square

Proposition 12. *The condition of the equation in the theorem (9) is equivalent to the fact one branch of the intersection of the sphere $\Sigma_{c(t), \dot{c}(t)}$ has at the point the same geodesic curvature as the curve C .*

Proof: As the sphere $\Sigma_{c(t), \dot{c}(t)}$ is osculating the curve C when it is a Darboux curve, the contact order with the curve should be at least 3. \square

When the direction ℓ defined by $\dot{c}(t)$ is not a principal direction, there is a unique sphere tangent to M which has contact with C of best order, the other having contact of order one with C . There is only one such sphere, in the pencil containing the osculating circle to C which is tangent to M . This last sphere should therefore be $\Sigma_{c(t), \dot{c}(t)}$

Remark: The previous considerations imply that the geodesic curvature of the branch of $S_\alpha \cap M$ tangent to the Darboux curve is equal to: $k_g = \frac{-k'_n}{\tau_g}$.

4 Darboux curves in cyclides: a geometric viewpoint

We can explicit $V(M)$ when the surface M is a regular Dupin cyclide (see Figure 9). It is the wedge of the two circles formed by the osculating spheres of the regular cyclide.

Consider a surface M with principal foliations \mathcal{P}_1 and \mathcal{P}_2 and umbilic set \mathcal{U} . The triple $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{U})$ will be referred as the em principal configuration of the surface.

Definition 13. *For each angle $\alpha \in (-\pi/2, \pi/2)$ we can consider the foliations \mathcal{F}_α^+ and \mathcal{F}_α^- such the leaves of this foliation are the curves making a constant angle $\pm\alpha$ with the leaves of the principal foliation \mathcal{P}_1 .*

In other words, the normal curvature of a leaf of \mathcal{F}_α is precisely $k_n(\alpha) = k_1 \cos^2 \alpha + k_2 \sin^2 \alpha$.

Let us, for later use, define the surfaces $M_\alpha \subset V(M)$ as the set of spheres tangent to M at a point m having curvature $k_\alpha = k_n(\alpha) = k_1 \cos^2 \alpha + k_2 \sin^2 \alpha$. Each surface M_α is foliated by the lifts of the curves of \mathcal{F}_α ; we call this new foliation $\tilde{\mathcal{F}}_\alpha$. These later foliation form a foliation of $V(M)$ that we call $\tilde{\mathcal{F}}$.

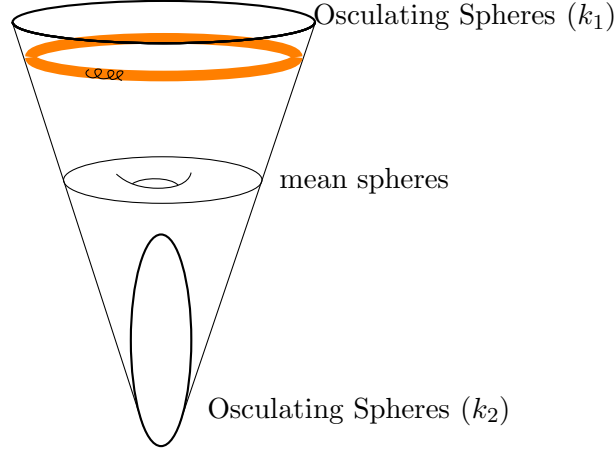


Figure 9: Darboux curves in $V(M)$ when M is a regular Dupin cyclide

Proposition 14. *The Darboux curves in Dupin cyclides are the leaves of the foliations \mathcal{F}_α^+ and \mathcal{F}_α^- .*

The proof of Proposition 14 is given in subsections 4.0.1 and 4.0.2.

4.0.1 Cylinders in \mathbb{R}^3

This is the easiest case to visualize. Cylinders are models for cyclides having exactly one singular point.

The definition of a Darboux curve requires that the osculating sphere to the helix \mathcal{H} drawn on the revolution cylinder is tangent to it. This comes from the fact that the helix is invariant by the rotation R of angle π and axis the principal curvature vector of the helix which is a vector orthogonal to the cylinder and going through its axis of revolution. As the osculating sphere should also be invariant by this rotation its diameter is contained in the line normal to the cylinder which is the axis of the rotation. Therefore the sphere is tangent to the cylinder.

4.0.2 Regular cyclides in \mathbf{S}^3

Each regular Dupin cyclide M is, for a suitable metric of constant curvature 1, the tubular neighborhood of a geodesic \mathcal{G}_1 of \mathbf{S}^3 (see [La-Wa]). Seeing \mathbf{S}^3 as the unit sphere of an euclidian space \mathbb{E}^4 of dimension 4, it is the intersection of a 2-plane P_1 with \mathbf{S}^3 . Let P_2 be the plane orthogonal to P_1 in \mathbb{E}^4 . Then it is also a tubular neighborhood of the geodesic $\mathcal{G}_2 = P_2 \cap \mathbf{S}^3$. We can define (for this metric) the symmetries with respect to the two spheres containing respectively m and \mathcal{G}_1 and m and \mathcal{G}_2 . Let \mathcal{R} be the composition of these symmetries. Then \mathcal{R} preserves the cyclide but also the “helices” on the cyclide. It should therefore also preserve the osculating sphere to the helix, which, as it is not normal to the cyclide, has to be tangent to it. The “helix” is therefore a Darboux curve. There are enough “helices” to be sure we got all the Darboux curves.

The last family of Dupin cyclides is formed of conformal images of cones of revolution (see [La-Wa]). On can deal cones of revolution as we did with regular cyclides, using a sphere

orthogonal to the axis and m (it belongs to the pencil with limit points the singular points) and a sphere containing the axis and m .

We will, using the general dynamical properties of Darboux curves, prove (see Proposition 5) that a surface is a Dupin cyclide if and only if its Darboux curves are the leaves of the foliation $\tilde{\mathcal{F}}$ of $V(M)$.

5 Differential Equation of Darboux Curves in a Principal Chart

Consider a local principal chart (u, v) in a surface $\mathbb{M} \subset \mathbb{R}^3$. The first and second fundamental forms are denoted by

$$I = Edu^2 + Gdv^2, \quad II = I = edu^2 + gdv^2$$

and the principal curvatures are $k_1 = e/E$ and $k_2 = g/G$.

Proposition 15. *Let (u, v) be a principal chart. Let c be a curve parametrized by arc length s making an angle $\alpha(s)$ with principal direction $(1, 0)$. For a Darboux line c the following differential equation is verified*

$$\begin{aligned} 3(k_1 - k_2) \sin \alpha \cos \alpha \frac{d\alpha}{ds} &= \frac{1}{\sqrt{E}} \frac{\partial k_1}{\partial u} \cos^3 \alpha + \frac{1}{\sqrt{G}} \frac{\partial k_2}{\partial v} \sin^3 \alpha \\ &= \frac{\partial k_1}{\partial s_1} \cos^3 \alpha + \frac{\partial k_2}{\partial s_2} \sin^3 \alpha. \end{aligned} \quad (10)$$

Here $(u', v') = (\frac{\cos \alpha}{\sqrt{E}}, \frac{\sin \alpha}{\sqrt{G}})$.

Proof. Consider a principal chart (u, v) such that $v = cte$ are the leaves of the principal foliation \mathcal{P}_1 .

Let $c(s) = (u(s), v(s))$ be a regular curve parametrized by arc length s . So we can write $c'(s) = (u', v') = (\frac{\cos \alpha}{\sqrt{E}}, \frac{\sin \alpha}{\sqrt{G}})$, defining a direction α with respect to principal foliation \mathcal{P}_1 , horizontal foliation.

We have the following classical relations:

$$\begin{aligned} k_n(\alpha) &= k_1 \cos^2 \alpha + k_2 \sin^2 \alpha \\ k_g &= \frac{d\alpha}{ds} + k_g^1 \cos \alpha + k_g^2 \sin \alpha \\ \tau_g &= (k_2 - k_1) \cos \alpha \sin \alpha. \end{aligned}$$

Here $k_g^1 = (k_g)_{v=const}$ and $k_g^2 = (k_g)_{u=const}$ are the geodesic curvatures of the coordinates curves.

Therefore,

$$\begin{aligned} \frac{dk_n}{ds} &= \frac{1}{\sqrt{E}} \frac{\partial k_1}{\partial u} \cos^3 \alpha + \frac{1}{\sqrt{G}} \frac{\partial k_1}{\partial v} \cos^2 \alpha \sin \alpha + \frac{1}{\sqrt{E}} \frac{\partial k_2}{\partial u} \cos \alpha \sin^2 \alpha \\ &\quad + \frac{1}{\sqrt{G}} \frac{\partial k_2}{\partial v} \sin^3 \alpha + 2(k_2 - k_1) \cos \alpha \sin \alpha \frac{d\alpha}{ds} \end{aligned}$$

The differential equation of Darboux curves is given by $k'_n + k_g \tau_g = 0$ and so it follows that:

$$\begin{aligned}
& [k_g^1(k_2 - k_1) + \frac{1}{\sqrt{G}} \frac{\partial k_1}{\partial v}] \cos^2 \alpha \sin \alpha + [k_g^2(k_2 - k_1) + \frac{1}{\sqrt{E}} \frac{\partial k_2}{\partial u}] \cos \alpha \sin^2 \alpha \\
& + 3(k_2 - k_1) \cos \alpha \sin \alpha \frac{d\alpha}{ds} + \frac{1}{\sqrt{E}} \frac{\partial k_1}{\partial u} \cos^3 \alpha + \frac{1}{\sqrt{G}} \frac{\partial k_2}{\partial v} \sin^3 \alpha = 0
\end{aligned}$$

In any orthogonal chart ($F = 0$) we have that $G_u = 2G\sqrt{E}k_g^2 = (k_g)_{u=cte}$ and $E_v = -2E\sqrt{G}k_g^1 = (k_g)_{v=cte}$. Also the Codazzi equations in a principal chart are given by:

$$\frac{\partial k_1}{\partial v} = \frac{E_v}{2E}(k_2 - k_1), \quad \frac{\partial k_2}{\partial u} = \frac{G_u}{2G}(k_1 - k_2).$$

See Struik book's [St, pages 113 and 120].

Therefore,

$$\begin{aligned}
k_g^1(k_2 - k_1) + \frac{1}{\sqrt{G}} \frac{\partial k_1}{\partial v} &= -\frac{E_v}{2E\sqrt{G}}(k_2 - k_1) + \frac{1}{\sqrt{G}} \frac{E_v}{2E}(k_2 - k_1) = 0 \\
k_g^2(k_2 - k_1) + \frac{1}{\sqrt{E}} \frac{\partial k_2}{\partial u} &= \frac{G_u}{2G\sqrt{E}}(k_2 - k_1) + \frac{1}{\sqrt{E}} \frac{G_u}{2G}(k_1 - k_2) = 0.
\end{aligned}$$

This ends the proof. \square

Remark: A curve $c(s)$ has contact of third order with the associated osculating sphere, tangent to the surface, when

$$\langle c', c' \rangle [2\langle N', c'' \rangle + \langle N'', c' \rangle] - 3\langle c', N' \rangle \langle c', c'' \rangle = 0.$$

This equation can be used to obtain the differential equation of Darboux curves in any chart (u, v) . See [Sa1].

6 A plane-field on $V(M)$

The two tangents to the two Darboux orbits through the point $(m, \alpha) \in V(M)$ define a plane in $T_{m, \alpha}V(M)$. Together, these planes form a plane-field \mathcal{P} .

Proposition 16. *The plane-field \mathcal{P} is integrable if and only if*

$$(\xi_1)\theta_2 = -\frac{1}{6}\theta_1\theta_2, \quad (\xi_2)\theta_1 = \frac{1}{6}\theta_1\theta_2.$$

Here ξ_i are the conformal vector fields and θ_i are the principal conformal curvatures.

Proof. Consider in the unitary tangent bundle the suspension of Darboux curves.

These curves are defined by the following vector field

$$\mathcal{D}_1 = \frac{\cos \alpha}{\sqrt{E}} \frac{\partial}{\partial u} + \frac{\sin \alpha}{\sqrt{G}} \frac{\partial}{\partial v} + \left[\frac{\partial k_1 / \partial u}{3\sqrt{E}(k_1 - k_2)} \frac{\cos^2 \alpha}{\sin \alpha} + \frac{\partial k_2 / \partial v}{3\sqrt{E}(k_1 - k_2)} \frac{\sin^2 \alpha}{\cos \alpha} \right] \frac{\partial}{\partial \alpha}.$$

Consider the involution $\varphi(u, v, \alpha) = (u, v, -\alpha)$ and the induced vector field $D_2 = \varphi_*(D_1)$.

So it follows that:

$$\mathcal{D}_2 = \frac{\cos \alpha}{\sqrt{E}} \frac{\partial}{\partial u} - \frac{\sin \alpha}{\sqrt{G}} \frac{\partial}{\partial v} - \left[-\frac{\partial k_1 / \partial u}{3\sqrt{E}(k_1 - k_2)} \frac{\cos^2 \alpha}{\sin \alpha} + \frac{\partial k_2 / \partial v}{3\sqrt{E}(k_1 - k_2)} \frac{\sin^2 \alpha}{\cos \alpha} \right] \frac{\partial}{\partial \alpha}.$$

Consider the plane field, *Darboux plane field*, defined by $\{\mathcal{D}_1, \mathcal{D}_2\}$.

In the sequence it will be obtained the conditions of integrability of this plane field.

In order to simplify the calculations the following changes will be developed.

First, consider the new pair of vector fields $\tilde{\mathcal{D}}_1 = \mathcal{D}_1 + \mathcal{D}_2$ and $\tilde{\mathcal{D}}_2 = \mathcal{D}_1 - \mathcal{D}_2$ and obtain:

$$\begin{aligned}\tilde{\mathcal{D}}_1 &= \frac{2 \cos \alpha}{\sqrt{E}} \frac{\partial}{\partial u} + \left[\frac{2\partial k_1/\partial u}{3\sqrt{E}(k_1 - k_2)} \frac{\cos^2 \alpha}{\sin \alpha} \right] \frac{\partial}{\partial \alpha} \\ \tilde{\mathcal{D}}_2 &= \frac{2 \sin \alpha}{\sqrt{G}} \frac{\partial}{\partial v} + \left[\frac{2\partial k_2/\partial v}{3\sqrt{E}(k_1 - k_2)} \frac{\sin^2 \alpha}{\cos \alpha} \right] \frac{\partial}{\partial \alpha}.\end{aligned}$$

Next consider:

$$\begin{aligned}\bar{\mathcal{D}}_1 &= \frac{2}{\sqrt{E}} \frac{\partial}{\partial u} + \left[\frac{2\partial k_1/\partial u}{3\sqrt{E}(k_1 - k_2)} \frac{\cos \alpha}{\sin \alpha} \right] \frac{\partial}{\partial \alpha} \\ \bar{\mathcal{D}}_2 &= \frac{2}{\sqrt{G}} \frac{\partial}{\partial v} + \left[\frac{2\partial k_2/\partial v}{3\sqrt{E}(k_1 - k_2)} \frac{\sin \alpha}{\cos \alpha} \right] \frac{\partial}{\partial \alpha}.\end{aligned}$$

Consider the unitary vector fields X_i , the conformal vector fields ξ_i and the principal conformal curvatures θ_i .

$$\begin{aligned}X_1 &= \frac{1}{\sqrt{E}} \frac{\partial}{\partial u}, & X_2 &= \frac{1}{\sqrt{G}} \frac{\partial}{\partial v} \\ \xi_1 &= \frac{2X_1}{k_1 - k_2}, & \xi_2 &= \frac{2X_2}{k_1 - k_2} \\ \theta_1 &= \frac{4(X_1)k_1}{(k_1 - k_2)^2}, & \theta_2 &= \frac{4(X_2)k_2}{(k_1 - k_2)^2}, \quad X_i(k_i) = Dk_i(X_i).\end{aligned}$$

Observing that $\frac{\partial k_1/\partial u}{\sqrt{E}} = (X_1)k_1$ and $\frac{\partial k_2/\partial v}{\sqrt{G}} = (X_2)k_2$ we obtain a new base defined by:

$$\begin{aligned}\mathcal{D}_1^c &= \xi_1 + \frac{1}{6}\theta_1 \frac{\cos \alpha}{\sin \alpha} \frac{\partial}{\partial \alpha} \\ \mathcal{D}_2^c &= \xi_2 + \frac{1}{6}\theta_2 \frac{\sin \alpha}{\cos \alpha} \frac{\partial}{\partial \alpha}.\end{aligned}$$

Recall that:

$$[fX, gY] = fg[X, Y] + (X.g)fY - (Y.f)gX$$

$$[\xi_1, \xi_2] = -\frac{1}{2}\theta_2 \xi_1 - \frac{1}{2}\theta_1 \xi_2.$$

So it follows that:

$$\begin{aligned}[\mathcal{D}_1^c, \mathcal{D}_2^c] &= [\xi_1, \xi_2] + \left[\xi_1, \frac{1}{6}\theta_2 \frac{\sin \alpha}{\cos \alpha} \frac{\partial}{\partial \alpha} \right] + \left[\frac{1}{6}\theta_1 \frac{\cos \alpha}{\sin \alpha} \frac{\partial}{\partial \alpha}, \xi_2 \right] + \left[\frac{1}{6}\theta_1 \frac{\cos \alpha}{\sin \alpha} \frac{\partial}{\partial \alpha}, \frac{1}{6}\theta_2 \frac{\sin \alpha}{\cos \alpha} \frac{\partial}{\partial \alpha} \right] \\ &= -\frac{1}{2}\theta_2 \xi_1 - \frac{1}{2}\theta_1 \xi_2 + \frac{1}{6}(\xi_1)\theta_2 \frac{\sin \alpha}{\cos \alpha} \frac{\partial}{\partial \alpha} - \frac{1}{6}(\xi_2)\theta_1 \frac{\cos \alpha}{\sin \alpha} \frac{\partial}{\partial \alpha} - \frac{1}{18} \frac{\theta_1\theta_2}{\sin \alpha \cos \alpha} \frac{\partial}{\partial \alpha} \\ &= -\frac{1}{2}\theta_2 \xi_1 - \frac{1}{2}\theta_1 \xi_2 + \frac{1}{6} \left[(\xi_1)\theta_2 \frac{\sin \alpha}{\cos \alpha} - (\xi_2)\theta_1 \frac{\cos \alpha}{\sin \alpha} - \frac{1}{3} \frac{\theta_1\theta_2}{\sin \alpha \cos \alpha} \right] \frac{\partial}{\partial \alpha}\end{aligned}$$

Now consider the determinant

$$\begin{vmatrix} 1 & 0 & \frac{1}{6}\theta_1 \frac{\cos \alpha}{\sin \alpha} \\ 0 & 1 & \frac{1}{6}\theta_2 \frac{\sin \alpha}{\cos \alpha} \\ -\frac{1}{2}\theta_2 & -\frac{1}{2}\theta_1 & \lambda \end{vmatrix}, \quad \lambda = \frac{1}{6}[(\xi_1)\theta_2 \frac{\sin \alpha}{\cos \alpha} - (\xi_2)\theta_1 \frac{\cos \alpha}{\sin \alpha} - \frac{1}{3} \frac{\theta_1 \theta_2}{\sin \alpha \cos \alpha}]$$

Evaluating this determinant it is obtained:

$$\frac{-3[(\xi_1)\theta_2 + (\xi_2)\theta_1] \cos 2\alpha + [3(\xi_1)\theta_2 - 3(\xi_2)\theta_1 + \theta_1 \theta_2]}{36 \sin \alpha \cos \alpha}$$

So the integrability conditions are given by:

$$(\xi_1)\theta_2 + (\xi_2)\theta_1 = 0, \quad 3(\xi_1)\theta_2 - 3(\xi_2)\theta_1 + \theta_1 \theta_2 = 0.$$

This is equivalent to: $(\xi_1)\theta_2 = -\frac{1}{6}\theta_1 \theta_2$ and $(\xi_2)\theta_1 = \frac{1}{6}\theta_1 \theta_2$. □

7 Darboux lines near a Ridge Point

Consider a surface M and a principal chart (u, v) such that the horizontal foliation \mathcal{P}_1 is that associated to the principal curvature k_1 .

Definition 17. A non umbilic point $p_0 = (u_0, v_0)$ is called a ridge point of the principal foliation \mathcal{P}_1 if $\frac{\partial k_1}{\partial u}(p_0) = 0$.

The ridges for the principal foliation \mathcal{P}_2 are characterized by $\frac{\partial k_2}{\partial v}(p_0) = 0$.

Definition 18. A ridge point relative to \mathcal{P}_1 is called zigzag, respectively beak to beak, when $\sigma_1(p_0) = \frac{\partial^2 k_1}{\partial u^2}(p_0)/(k_1(p_0) - k_2(p_0)) < 0$, respectively, $\sigma_1(p_0) > 0$.

A ridge point p_0 relative to \mathcal{P}_2 is called zigzag, respectively beak to beak, when $\sigma_2(p_0) = \frac{\partial^2 k_2}{\partial v^2}(p_0)/(k_2(p_0) - k_1(p_0)) < 0$, respectively $\sigma_2(p_0) > 0$.

The ridge points are associated to inflections of the principal curvature lines, to the singularities of the focal set of the surface and also with the singularities of the boundary of $V(M)$, the space of spheres tangent to M . See [Po] for an introduction to ridges and also [Gu] in his study of geometric optic and applications in construction of eye lens.

A practical way to see the type of a ridge point is given by the following proposition.

Proposition 19. Consider a surface of class C^r , $r \geq 4$ parametrized by the graph $(u, v, h(u, v))$ where

$$\begin{aligned} h(u, v) = & \frac{k_1}{2}u^2 + \frac{k_2}{2}v^2 + \frac{a}{6}u^3 + \frac{d}{2}u^2v + \frac{b}{2}uv^2 + \frac{c}{6}v^3 \\ & + \frac{A}{24}u^4 + \frac{B}{6}u^3v + \frac{C}{4}u^2v^2 + \frac{D}{6}uv^3 + \frac{E}{24}v^4 + h.o.t \end{aligned}$$

Then $(0, 0)$ is a ridge point for \mathcal{P}_1 when $a = 0$. Also $\sigma_1 = [\frac{A-3k_1^3}{k_1-k_2} + \frac{2d^2}{(k_1-k_2)^2}]$.

Corresponding to the foliation \mathcal{P}_2 the point $(0, 0)$ is a ridge point when $c = 0$. Also $\sigma_2(0) = [\frac{E-3k_2^3}{k_2-k_1} + \frac{2b^2}{(k_2-k_1)^2}]$.

Proof. Straightforward calculations shows that the principal curvatures in a neighborhood of $(0, 0)$ are given by:

$$\begin{aligned}
k_1(u, v) &= k_1 + au + dv + \frac{1}{2}\left(A - 3k_1^3 + \frac{2d^2}{k_1 - k_2}\right)u^2 + \left(B - 2\frac{bd}{k_2 - k_1}\right)uv \\
&\quad + \frac{1}{2}\left(C - k_1k_2^2 - \frac{2b^2}{k_2 - k_1}\right)v^2 + h.o.t. \\
k_2(u, v) &= k_2 + bu + cv + \frac{1}{2}\left(C - k_1^2k_2 + \frac{2d^2}{k_2 - k_1}\right)u^2 + \left(D + 2\frac{bd}{k_2 - k_1}\right)uv \\
&\quad + \frac{1}{2}\left(E - 3k_2^3 + \frac{2b^2}{k_2 - k_1}\right)v^2 + h.o.t.
\end{aligned} \tag{11}$$

So the result follows. \square

Proposition 20. *Let p_0 be a ridge point of M corresponding to principal foliation \mathcal{P}_1 such that $\sigma_1(p_0) \neq 0$. Then the ridge set R containing p_0 is locally a regular curve transversal to \mathcal{P}_1 and the boundary of $V(M)$ corresponding to $\alpha = 0$ has a cuspidal edge along $\pi^{-1}(R)$. See Fig. 10. Analogously for the ridges associated to the principal foliation \mathcal{P}_2 .*

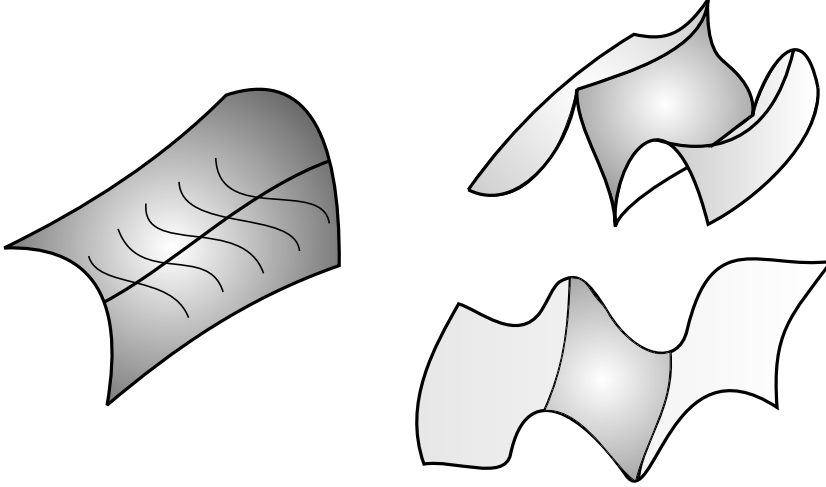


Figure 10: Ridges and Singularities of the boundary of $V(M)$

Proof. In the parametrization given in Proposition 19 it follows that the at $p_0 = (0, 0)$ the principal direction corresponding to \mathcal{P}_1 is $e_1 = (1, 0)$ and the ridge set is parametrized, according to equation (11), by: $(u(v), v) = \left(-\frac{d(k_1 - k_2)}{(A - 3k_1^3)(k_1 - k_2) + 2d^2}v + O(v^2), v\right)$.

Next consider a principal chart (u, v) . The set of spheres $V(M) \subset \Lambda^4$ is parametrized by $\sigma(u, v, \alpha) = k_n(\alpha)m(u, v) + N(u, v)$ with $k_n(\alpha) = k_1(u, v)\cos^2\alpha + k_2(u, v)\sin^2\alpha$, $\mathcal{L}(m, m) = 0$

and $N_u = -k_1 m_u$, $N_v = -k_2 m_v$. See equation (1). We have that

$$\begin{aligned}\sigma_u &= \left[\frac{\partial k_1}{\partial u} \cos^2 \alpha + \frac{\partial k_2}{\partial u} \sin^2 \alpha \right] m + (k_n - k_1) m_u \\ \sigma_v &= \left[\frac{\partial k_1}{\partial v} \cos^2 \alpha + \frac{\partial k_2}{\partial v} \sin^2 \alpha \right] m + (k_n - k_2) m_v \\ \sigma_\alpha &= [(k_2 - k_1) \cos \alpha \sin \alpha] m\end{aligned}$$

So $D\sigma$ has rank 3 for $\alpha \in (0, \frac{\pi}{2})$.

The boundary of $V(M)$ is parametrized by $\alpha = 0$ and $\alpha = \pi/2$ and so $\sigma_1(u, v) = k_1(u, v)m(u, v) + N(u, v)$ and $\sigma_2(u, v) = k_2(u, v)m(u, v) + N(u, v)$. The map σ_1 has rank 1 at the ridges and so we have the structure of cuspidal edges on the boundary of $V(M)$. \square

Proposition 21. *Let p_0 be a non umbilic and disjoint from the ridge set. The Darboux curves tangent to the principal lines are given by cuspidal curves $r_1(t) = (\frac{3}{2} \frac{\partial k_1}{\partial u} (k_1 - k_2) t^2 + \dots, \frac{\partial k_1}{\partial u} (k_1 - k_2) t^3 + \dots)$ and $r_2(t) = (\frac{\partial k_2}{\partial v} (k_2 - k_1) t^3 + \dots, \frac{3}{2} \frac{\partial k_2}{\partial v} (k_2 - k_1) t^2 + \dots)$. The behavior of all the Darboux curves passing through p_0 is as shown in the Fig. 11.*

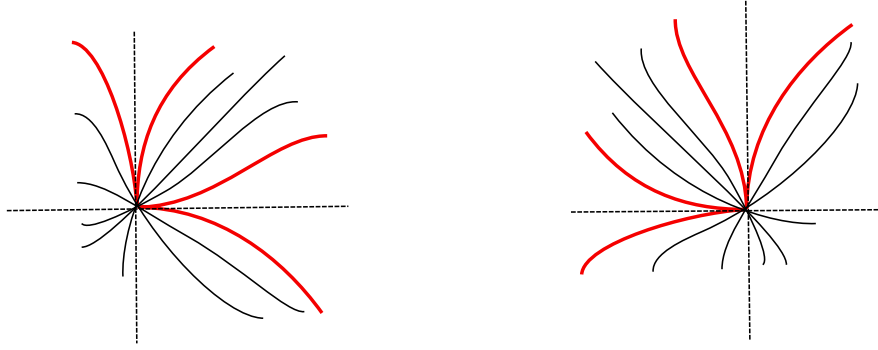


Figure 11: Darboux curves through a non ridge point

Proof. Consider the vector field X defined by the differential equation:

$$\begin{aligned}u' &= \frac{1}{\sqrt{E}} \cos \alpha [3(k_1 - k_2) \sin \alpha \cos \alpha] \\ v' &= \frac{1}{\sqrt{G}} \sin \alpha [3(k_1 - k_2) \sin \alpha \cos \alpha] \\ \alpha' &= \frac{1}{\sqrt{E}} \frac{\partial k_1}{\partial u} \cos^3 \alpha + \frac{1}{\sqrt{G}} \frac{\partial k_2}{\partial v} \sin^3 \alpha.\end{aligned}$$

The projections of the integral curves of X in the coordinates (u, v) are precisely the Darboux curves.

For any initial condition $(0, \alpha_0)$, with $\alpha_0 \neq k\pi/2$, the integral curves of X are transversal to the axis α and so has a regular projection. For $\alpha_0 = n\pi/2$ and $\sigma_2 \neq 0$, the integral curves of X are tangent to the axis α and the projections are of cuspidal type. For $\alpha = n\pi$ direct calculations gives: $(u(t), v(t)) = (\frac{3}{2} \frac{\partial k_1}{\partial u} (k_1 - k_2) t^2 + \dots, (-1)^n \frac{\partial k_1}{\partial u} (k_1 - k_2) t^3 + \dots)$. Now observe that the projection of the integral curves passing through $(0, 0, 0)$ and $(0, 0, \pi)$ are both tangent to semi axis of u . \square

Theorem 22. *Let R be an arc of ridge points transversal to the corresponding principal foliation, i.e., suppose that $\sigma_i(p) \neq 0$ for every $p \in R$. Then there are two types of behavior for the Darboux curves near the ridge set, the first is the zigzag and the other is the beak to beak.*

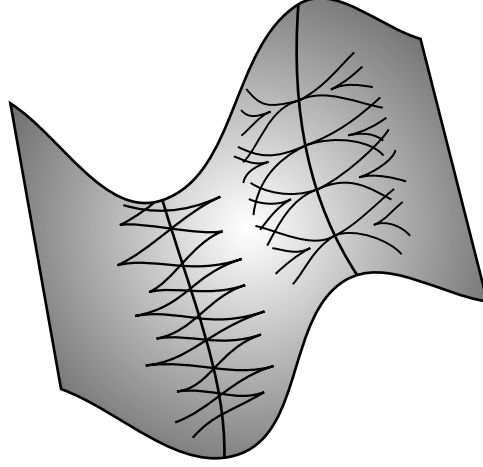


Figure 12: Darboux curves near regular curve of ridges: zigzag and beak to beak.

Proof. Consider the vector field X as in the proof of proposition 21.

We will consider only the ridge set corresponding to \mathcal{P}_1 . For the other principal foliation the analysis is similar. The ridge set is defined by the equations $\frac{\partial k_1}{\partial u}(u, v) = 0$ and $\frac{\partial k_2}{\partial v}(u, v) = 0$, one for the corresponding principal foliation.

The singularities of X are defined by $(U(v), v, 0)$ and $(u, V(u), \frac{\pi}{2})$, where $\frac{\partial k_1}{\partial u}(U(v), v) = 0$ and $\frac{\partial k_2}{\partial v}(u, V(u)) = 0$.

To simplify the notation suppose a singular point $(0, 0, 0)$ of the ridge transversal to the principal foliation \mathcal{P}_1 and $E(0) = G(0) = 1$.

It follows that:

$$DX(0) = \begin{pmatrix} 0 & 0 & 3(k_1 - k_2) \\ 0 & 0 & 0 \\ \frac{\partial^2 k_1}{\partial u^2} & 0 & 0 \end{pmatrix}$$

The eigenvalues of $DX(0)$ are:

$$\lambda_1 = 0, \lambda_2 = \frac{1}{\sqrt{3}} \sqrt{\frac{\partial^2 k_1}{\partial u^2} / (k_1 - k_2)}, \lambda_3 = -\frac{1}{\sqrt{3}} \sqrt{\frac{\partial^2 k_1}{\partial u^2} / (k_1 - k_2)}.$$

By invariant manifold theory, when $\lambda_2 \lambda_3 = -\frac{1}{3} \frac{\partial^2 k_1}{\partial u^2} / (k_1 - k_2) = -\frac{1}{3} \sigma_1(0) < 0$, the singular set of X (ridge set) is normally hyperbolic and there are stable and unstable surfaces, normally hyperbolic along the singular set. This implies that there is a lamination (continuous fibration) along the ridge set and the fibers are the Darboux curves. Also the prolonged Darboux curves are of class C^1 along the ridge set.

So the Darboux curves are as shown in Fig. 12, center and right. That is, there are Darboux curve crossing the ridge, tangent to the principal lines, and the prolonged Darboux curves are C^1 along the ridge set.

In the case when $\sigma_1(0) = \frac{\partial^2 k_1}{\partial u^2} / (k_1 - k_2) < 0$ the non zeros eigenvalues of $DX(0)$ are purely complex and so the singular set is not normally hyperbolic.

In this case we are in the hypothesis of Roussarie Theorem, [Ro, Theorem 20, page 59], so there is a local first integral in a neighborhood of the ridge set. The level sets of this first integral are cylinders and the integral curves (helices) in each cylinder when projected in the surface M has a cuspidal point exactly when helix cross the section $\alpha = 0$. This produces the zigzag.

There are no Darboux curves tangent to the principal direction e_1 along the ridge set in this case. \square

8 Darboux curves in general cylinders and cones and in revolution surfaces

Darboux curves on general cones where already studied by Santaló ([Sa2]) In a similar way, one can study Darboux curves on cylinders and surfaces of revolution. This is not a coincidence. The three type of surfaces are canal surfaces corresponding to a curve $\gamma \subset \Lambda^4$ which is also contained in a 3-dimensional subset of \mathbb{L}^5 . Depending on the subspace, this intersection is either a copy of Λ^2 , a unit sphere \mathbf{S}^3 or a 2-dimensional cylinder (see [Da1], [M-N], [Ba-La-Wa]). The latter condition defines conformal images of general cones, general cylinders and surfaces of revolution.

These surfaces can be obtained imposing conformally invariant local conditions.

Recall that, assuming that S is a surface which is *umbilic free*, that is, that the principal curvatures $k_1(x)$ and $k_2(x)$ of S are different at any point x of S . Let X_1 and X_2 be unit vector fields tangent to the curvature lines corresponding to, respectively, k_1 and k_2 . Throughout the paper, we assume that $k_1 > k_2$. Put $\mu = (k_1 - k_2)/2$. Since more than 100 years, it is known ([Tr], see also [CSW]) that the vector fields $\xi_i = X_i/\mu$ and the coefficients θ_i ($i = 1, 2$) in

$$[\xi_1, \xi_2] = -\frac{1}{2}(\theta_2 \xi_1 + \theta_1 \xi_2)$$

are invariant under arbitrary (orientation preserving) conformal transformation of \mathbb{R}^3 . (In fact, they are invariant under arbitrary conformal change of the Riemannian metric on the ambient space). Elementary calculation involving Codazzi equations shows that

$$\theta_1 = \frac{1}{\mu^2} \cdot X_1(k_1) \quad \text{and} \quad \theta_2 = \frac{1}{\mu^2} \cdot X_2(k_2).$$

The quantities θ_i ($i = 1, 2$) are called *conformal principal curvatures* of S .

Canal surface are characterized locally, [Da1], [H-J], [M-N], [Ba-La-Wa], by the following propositions.

Proposition 23. *A surface S is (a piece of) of a canal if and only if one of its conformal principal curvatures, say θ_2 , is equal to zero.*

Proposition 24. *Imposing moreover that the other conformal curvature, say θ_1 , is constant along characteristic circles characterizes the surface is one of the three family above (cone, cylinder and surfaces of revolution).*

Proposition 25. Let M be a surface and (u, v) be a principal chart such that $\theta_1(u, v) = \theta_1(u)$ and $\theta_2(u, v) = 0$. Let $A(u) = \exp[\int \frac{k'_1}{k_1 - k_2} du]$ and $\alpha \in (0, \pi)$ be an angle. Then the function $\mathcal{J}(u, \alpha) = A(u) \cos^3 \alpha$ is a first integral of the Darboux curves. Moreover in the region $A_c = \pi(M_c) = \{(u, v) : u \in M_c\}$, $M_c = \mathcal{J}^{-1}(c)$, the Darboux curves are defined by the implicit differential equation

$$c^{2/3} G dv^2 - E(A^{2/3} - c^{2/3}) du^2 = 0.$$

Proof. The differential equation (10) is simplified to

$$u' = \frac{\cos \alpha}{\sqrt{E}}, \quad v' = \frac{\sin \alpha}{\sqrt{G}}, \quad \alpha' = \frac{1}{3\sqrt{E}} \frac{k'_1}{k_1 - k_2} \frac{\cos^2 \alpha}{\sin \alpha}.$$

So it follows that $\frac{d\alpha}{du} = \frac{1}{3} \frac{k'_1}{k_1 - k_2} \frac{\cos \alpha}{\sin \alpha}$ which is a equation of separable variables. Direct integration leads to the first integral \mathcal{J} as stated.

To obtain the implicit differential equation solve the equation $\mathcal{J}(u, v) = c$ in function of $\cos \alpha$ and observe that $\frac{dv}{du} = \frac{\sqrt{E} \sin \alpha}{\sqrt{G} \cos \alpha}$. \square

Proposition 26. Let M be a surface and (u, v) be a principal chart such that $\theta_1(u, v) = \theta_1(u)$ and $\theta_2(u, v) = 0$. Then the plane-field \mathcal{P} is integrable.

Proof. Direct from the characterization of integrability of \mathcal{P} established in Proposition 16. \square

8.1 Darboux lines on surfaces of Revolution as Canal Surface

Consider an one parameter family of spheres of radius $r(u)$ with center at $(0, 0, u)$.

The envelope of this family is a canal surface and can be parametrized by:

$$H(u, v) = r(u) \cos \beta(u) (\cos v, \sin v, 0) + (0, 0, u - r(u) \sin \beta(u)),$$

where $\cos \beta(u) = \sqrt{1 - r'(u)^2}$, $\sin \beta = r'$, $|r'(u)| < 1$, $\beta \in (-\pi/2, \pi/2)$.

The normal unitary to the surface is:

$$N = (-\cos \beta(u) \cos v, -\cos \beta(u) \sin v, \sin \beta(u)).$$

The coefficients of the first and second fundamental forms of H are given by:

$$E(u, v) = \frac{(1 - r'^2 - rr'')^2}{1 - r'^2}, \quad F(u, v) = 0, \quad G(u, v) = r^2(1 - r'^2)$$

$$e(u, v) = -\frac{r''(1 - r'^2 - rr'')}{1 - r'^2}, \quad f(u, v) = 0, \quad g(u, v) = r(1 - r'^2).$$

The principal curvatures are given by:

$$k_1(u, v) = -\frac{r''}{1 - r'^2 - rr''} \quad k_2(u, v) = \frac{1}{r}.$$

It will be assumed that $k_2 > k_1$ and the surface is free of umbilic points.

The ridge set is defined by the equation

$$R(u, v) = \frac{\partial k_1}{\partial u} = r'''(1 - r'^2) + 3r'r''^2 = 0.$$

Proposition 27. *The Darboux lines on surfaces of revolution, free of umbilic points, can be integrated by quadratures. The function*

$$\mathcal{I}(u, \alpha) = r \cos \beta (k_1 - k_2) \cos^3 \alpha = h(u) \cos^3 \alpha,$$

is a first integral of the differential equation of Darboux lines. Here h is the distance of the point of the surface to the axis of revolution.

Moreover, if $R'(u) < 0$ the ridge is zigzag. If $R'(u) > 0$ the ridge is beak to beak. See Fig. 13.

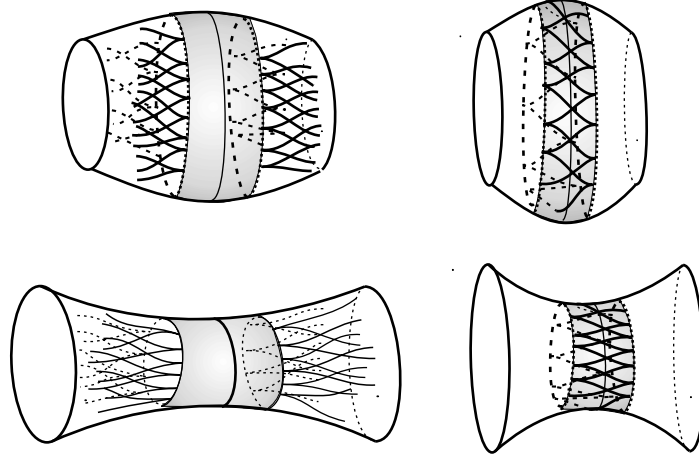


Figure 13: Darboux lines near regular ridges on surfaces of revolution

Proof. In the principal chart (u, v) the differential equation of Darboux lines is given by

$$u' = \frac{1}{\sqrt{E}} \cos \alpha, \quad v' = \frac{1}{\sqrt{G}} \sin \alpha, \quad \alpha' = \frac{1}{3\sqrt{E}} \left[\frac{k_1'}{k_1 - k_2} \right] \frac{\sin^2 \alpha}{\cos \alpha}$$

So it follows that $3 \frac{\sin \alpha}{\cos \alpha} d\alpha = \frac{k_1'}{k_1 - k_2} du$.

Now observe that

$$\begin{aligned} \int \frac{k_1'}{k_1 - k_2} du &= \int \frac{k_1' - k_2'}{k_1 - k_2} du + \int \frac{k_2'}{k_1 - k_2} du = \ln(k_2 - k_1) + \int \frac{k_2'}{k_1 - k_2} du \\ \int \frac{k_2'}{k_1 - k_2} du &= \int \left[\frac{r'}{r} + \frac{r' r''}{1 - r'^2} \right] du = \ln r (1 - r'^2)^{1/2} \end{aligned}$$

Therefore it follows that

$$\mathcal{I}(u, v, \alpha) = r(1 - r'^2)^{1/2} (k_1 - k_2) \cos^3 \alpha = r \cos \beta (k_1 - k_2) \cos^3 \alpha = h(u) (k_1 - k_2) \cos^3 \alpha$$

is the first integral. The analysis of behavior near ridges follows from Theorem 22. \square

Proposition 28. *The Darboux lines on a cone, free of umbilic points, can be integrated by quadratures. The function*

$$\mathcal{I}(u, \alpha) = k_g(u) \cos^3 \alpha$$

is a first integral of the differential equation of Darboux lines. Here k_g is the geodesic curvature of the intersection of the cone with the unitary sphere.

Moreover, if $k'_g/k_g < 0$ the ridge is zigzag. If $k'_g/k_g > 0$ the ridge is beak to beak. See Fig. 14.

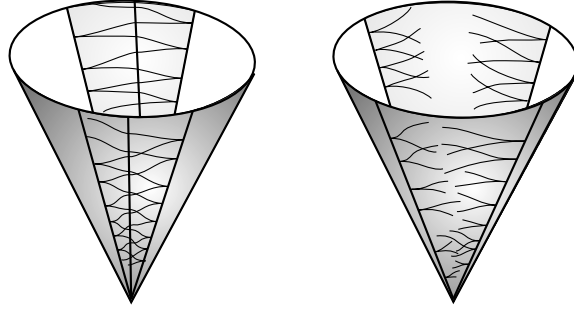


Figure 14: Darboux lines near regular ridges on a general cone

Proof. The cone can be parametrized by $X(u, v) = v\gamma(u)$ where $|\gamma| = 1$ and $|\gamma'| = 1$ is a spherical curve. We have that $k_1(u, v) = k_g(u)$ and $k_2(u, v) = 0$, since $\gamma'' = -\gamma + k_g\gamma \wedge \gamma'$ and $N(u, v) = \gamma \wedge \gamma'$.

The Darboux curves are given by: $(\sin \alpha / \cos \alpha) d\alpha = \frac{1}{3}(k'_g/k_g) du$. So it follows that $\mathcal{I}(u, \alpha) = k_g(u) \cos^3 \alpha$ is the first integral. The analysis of behavior near ridges follows from Theorem 22. \square

Proposition 29. *The ridges on a surface of revolution are characterized by the singularities of the curve $\mathcal{O}_1 \subset \Lambda^2 \subset \Lambda^4$ defined by the osculating circles of envelope of plane curves with center $(0, u)$ and radius $r(s)$.*

Moreover the ridges correspond to “vertices” of the curve $\gamma \subset \Lambda^2 \subset \Lambda^4$ defining the surface of revolution as an envelope.

Proof. The envelope of plane curves with center $(0, u)$ and radius $r(u)$ is a plane curve $c(u)$ with curvature given by $k(u) = -\frac{r''}{1-r'^2-rr''}$.

According to equation (1) the curve $\mathcal{O}_1(u) = k(u)m(u) + n(u)$ with $\mathcal{L}(m, m) = 0$, $\mathcal{L}(n, n) = 1$ and $n'(u) = -k(u)m'(u)$. So it follows that $\mathcal{O}'_1 = k'(u)m(u)$ is a light vector and is zero precisely at the points where $k'(u) = 0$.

Let us remark that if a point of $\gamma(s) \in \gamma$ is a vertex then the radius $\rho(s)$ of the osculating circle $\mathcal{O}s$ is critical. Let us denote by $P(s)$ the osculating plane to γ at $\gamma(s)$, and by $x(s) \in P(s)$ the center of the osculating plane, which is also the (unique) critical point of the restriction of \mathcal{L} to $P(s)$. The Dupin necklace formed by “the other osculating spheres” along the characteristic circle $C(s) \subset \Sigma(s)$, where the sphere $\Sigma(s)$ correspond to the point $\gamma(s) \in \Lambda^4$ is the envelope of the spheres of the circle $\mathcal{O}s$ and of the sphere of the associated circle \mathcal{O}^*s . This circle is contained in the affine plane $Q(s)$, orthogonal to $\text{span}(O, P(s))$ and which contains the point $y(s)$ which belongs to the line spanned by $x(s)$ and satisfies $\mathcal{L}(x(s), y(s)) = 1$. The square

Lorentz norm $\mathcal{L}(y(s_0))$ is critical when $\mathcal{L}(x(s_0))$ and $\rho(s_0)$ are. Let $\sigma(s)$ be a curve of osculating spheres of the other family along a line principal curvature. It is a light-like curve. Notice that, as $Q(s)$ is orthogonal to $y(s)$, $\mathcal{L}(y(s), \sigma(s)) = \mathcal{L}(y(s))$. Derivating $\mathcal{L}(y(s), \sigma(s))$ with respect to s we get, for a critical s_0 , $\mathcal{L}(y(s), \frac{d\sigma}{ds})|_{s=s_0} + \mathcal{L}(\frac{dy}{ds}, \sigma(s))|_{s=s_0} = 0$. As $y(s)$ belongs to the 3-dimensional space $A = \text{span}(\mathcal{O}s, O)$, which is independent of s , $\frac{dy}{ds}$ belongs also to A . All the affine spaces $Q(s)$ are orthogonal to A , therefore $\mathcal{L}(\frac{dy}{ds}, \sigma(s))|_{s=s_0} = \mathcal{L}(\frac{dy}{ds}, y(s))|_{s=s_0} = 0$. Therefore $\mathcal{L}(y(s), \frac{d\sigma}{ds})|_{s=s_0} = 0$. This is possible if and only if $\frac{d\sigma}{ds}|_{s=s_0} = 0$, as a Lorentz scalar product of non-zero time-like and light-like vector cannot be zero. The sphere σ_0 is therefore a ridge sphere. This reasoning is valid for all the curves in Λ^4 formed of osculating spheres of the other family along a line of principal curvature. The whole characteristic circle $C(s_0)$ is therefore a ridge. \square

Remark: Similarly in a cylinder $\alpha(u, v) = c(u) + v\vec{z}$, where c is a plane curve with curvature k the function $\mathcal{I}(u, \alpha) = k(u) \cos^3 \alpha$ is a first integral of Darboux curves.

Remark: The geodesics on surfaces of revolution has a first integral given by $\mathcal{J}(u, \alpha) = h(u) \sin \alpha$. The parallel $u = u_0$ is a geodesic if and only if $h'(u_0) = 0$, and it is a hyperbolic geodesic when $h''(u_0) > 0$. In a cone a first integral for the geodesics is given by $\mathcal{J}(v, \alpha) = \cos \alpha/v$.

9 Darboux curves in quadrics

The Darboux curves in the ellipsoid were considered in [Pe] by Pell. Here we complete and hopefully simplify his work.

The quadrics $\mathbb{Q}_{a,b,c}$ belongs to the triple orthogonal system of surfaces defined by the one parameter family of quadrics, $\frac{x^2}{a-\lambda} + \frac{y^2}{b-\lambda} + \frac{z^2}{c-\lambda} = 1$ with $a > b > c > 0$, see also [Sp] and [St].

Consider the principal chart (u, v) and the parametrization of $\mathbb{Q}_{a,b,c}$ given by equation (2).

For the ellipsoid $u \in (b, a)$, $v \in (c, b)$ or $u \in (c, b)$, $v \in (b, a)$.

For the hyperboloid of one sheet $u \in (b, a)$, $v < c$ or $u < c$, $v \in (b, a)$.

For the hyperboloid of two sheets $u \in (c, b)$, $v < c$ or $u < c$, $v \in (c, b)$.

The first fundamental form of $\mathbb{Q}_{a,b,c}$ is given by equation (3) and the second is given by equation (4).

Therefore the principal curvatures are given by:

$$k_1 = \frac{e}{E} = \frac{1}{u} \sqrt{\frac{abc}{uv}}, \quad k_2 = \frac{g}{G} = \frac{1}{v} \sqrt{\frac{abc}{uv}}.$$

Also consider the functions

$$r_1 = \frac{\partial k_1}{\partial u} / (3(k_1 - k_2)) = \frac{1}{2} \frac{v}{u(u-v)}, \quad r_2 = \frac{\partial k_2}{\partial v} / (3(k_1 - k_2)) = \frac{1}{2} \frac{u}{v(u-v)}.$$

The four umbilic points are $(\pm x_0, 0, \pm z_0) = (\pm \sqrt{\frac{a(a-b)}{a-c}}, 0, \pm \sqrt{\frac{c(b-c)}{a-c}})$.

Proposition 30. *The differential equation of Darboux curves on a quadric $\mathbb{Q}_{a,b,c}$ is given by:*

$$\begin{aligned}
u' &= \frac{1}{\sqrt{E}} \cos^2 \alpha \sin \alpha, & v' &= \frac{1}{\sqrt{G}} \cos \alpha \sin^2 \alpha \\
\alpha' &= \frac{r_1}{\sqrt{E}} \cos^3 \alpha + \frac{r_2}{\sqrt{G}} \sin^3 \alpha
\end{aligned} \tag{12}$$

Proof. In a principal chart the differential equation of Darboux curves is given by equation (10). Define $\tan \alpha = \frac{\sqrt{G}v'}{\sqrt{E}u'}$ so that $Eu'^2 + Gv'^2 = 1$. To obtain a regular extension of the differential equation (10) to $\alpha = m\pi/2$ and consider it as a vector field in the variables (u, v, α) multiply the resultant equation by the factor $\cos \alpha \sin \alpha$ and the result follows. \square

Proposition 31. *The function*

$$I(u, v, u', v') = I(u, v, \alpha) = \frac{\cos^2 \alpha}{u} + \frac{\sin^2 \alpha}{v}, \quad \tan \alpha = \frac{v'}{u'} \sqrt{\frac{G(u, v)}{E(u, v)}} \tag{13}$$

is a first integral of equation (12).

Proof. We have that $I(u, v, \alpha) = \frac{\cos^2 \alpha}{u} + \frac{\sin^2 \alpha}{v}$.

We will show that $\frac{d}{ds}(I(u(s), v(s), \alpha(s))) = 0$ along a solution $(u(s), v(s), \alpha(s))$.

We have that

$$\begin{aligned}
E_s &= \frac{1}{4} \frac{uv'}{H(u)} - \frac{1}{4} \frac{u'}{H(u)^2} [(u - 2v)H(u) + (uv - u^2)H'(u)] \\
G_s &= \frac{1}{4} \frac{vu'}{H(v)} - \frac{1}{4} \frac{v'}{H(v)^2} [(u - 2v)H(v) + (v^2 - uv)H'(v)]
\end{aligned}$$

Straightforward calculation leads to $\frac{d}{ds}(I(u(s), v(s), u'(s), v'(s))) = 0$. \square

Proposition 32. *The Darboux curves on a quadric $\mathbb{Q}_{a,b,c}$ are the real integral curves of the implicit differential equation:*

$$(v - \lambda)H(u)v'^2 - (u - \lambda)H(v)u'^2 = 0.$$

The normal curvature in a Darboux direction \mathcal{D} defined by $I(u, v, dv/du) = 1/\lambda$ is given by $k_n(p, \mathcal{D}) = \frac{1}{\lambda} \sqrt{\frac{abc}{uv}}$.

This differential equation is equivalent to

$$k_n(u, v, [du : dv]) = \frac{e(u, v)du^2 + g(u, v)dv^2}{E(u, v)du^2 + G(u, v)dv^2} = \frac{1}{\lambda} \sqrt{\frac{abc}{uv}} = \frac{1}{\lambda} (abc)^{1/4} \mathcal{K}^{1/4}.$$

Proof. From the equation $I = 1/\lambda$ it follows that

$$\left(\frac{dv}{du}\right)^2 = \frac{(u - \lambda) v E(u, v)}{(v - \lambda) u G(u, v)} = \frac{(u - \lambda) H(v)}{(v - \lambda) H(u)}.$$

Here dv/du is a Darboux direction \mathcal{D} .

As $k_n = \frac{e+g(dv/du)^2}{E+G(dv/du)^2}$ it follows from the equation above and from equations (3) and (4) that $k_n(p, \mathcal{D}) = \frac{1}{\lambda} \sqrt{\frac{abc}{uv}} = \frac{1}{\lambda} (abc)^{1/4} \mathcal{K}^{1/4}$, $\mathcal{K} = k_1 k_2$.

For the reciprocal part consider the implicit differential equation

$$\frac{e(u, v)du^2 + g(u, v)dv^2}{E(u, v)du^2 + G(u, v)dv^2} = \frac{1}{\lambda} \sqrt{\frac{abc}{uv}}.$$

Using equations (3) and (4) it follows that this equation is equivalent to the following.

$$\frac{\lambda - u}{H(u)} du^2 - \frac{\lambda - v}{H(v)} dv^2 = 0 \Leftrightarrow (u - \lambda)H(v)du^2 - (v - \lambda)H(u)dv^2 = 0.$$

The restriction in the values of λ is in order to consider only real solutions of the implicit differential equation obtained. \square

Proposition 33. *The ridge set of the quadric $\mathbb{Q}_{a,b,c}$ is the intersection of the quadric with the coordinates planes. Moreover:*

a) *For the ellipsoid with $0 < c < b < a$ it follows that:*

- i) *The ellipse $E_{xy} = \{z = 0\} \cap \mathbb{E}_{a,b,c}$, respectively $E_{yz} = \{x = 0\} \cap \mathbb{E}_{a,b,c}$, is a ridge corresponding to k_2 , respectively to k_1 , and is zigzag.*
- ii) *The ellipse $E_{xz} = \{y = 0\} \cap \mathbb{E}_{a,b,c}$ containing the four umbilic points $(\pm x_0, 0, \pm z_0)$ is the union of ridges of k_1 and k_2 . For $|x| > x_0$ the ridge correspond to k_1 . The ellipse E_{xz} is beak to beak in both cases.*

b) *For the hyperboloid of one sheet with $c < 0 < b < a$ it follows that:*

- i) *The hyperbole H_{yz} is a ridge corresponding to k_1 and is beak to beak.*
- ii) *The hyperbole H_{xz} is a ridge corresponding to k_1 and is zigzag.*
- iii) *The ellipse E_{xy} is a ridge corresponding k_2 and it is zigzag.*

c) *For the hyperboloid of two sheets with $c < b < 0 < a$ it follows that:*

- i) *The hyperbole H_{xz} is a ridge corresponding to k_1 and is zigzag.*
- ii) *The hyperbole H_{xy} containing the four umbilic points $(\pm x_0, \pm y_0, 0)$ is the union of ridges of k_1 and k_2 . For $|x| > x_0$ the ridge correspond to k_1 and all segments of hyperbolas are beak to beak.*

Proof. a) Ellipsoid: By symmetry is clear that the points of intersection of the coordinates planes with the ellipsoid are ridges points. The principal curvatures have no critical points along the corresponding principal curvature line in the complement of these three ellipses. In a principal chart (u, v) we have that $\frac{dk_1}{du} = -\frac{3}{2u}k_1 \neq 0$ and $\frac{dk_2}{dv} = -\frac{3}{2v}k_2 \neq 0$.

It will be sufficient to check the condition of zigzag or beak to beak only in a point of a connected component of the ridge set.

Consider the point $p_0 = (-\sqrt{a}, 0, 0)$. The ellipsoid is parametrized by:

$$x(y, z) = -\sqrt{a} + \sqrt{a} \left[\frac{y^2}{2b} + \frac{z^2}{2c} + \frac{y^4}{8b^2} + \frac{y^2 z^2}{4bc} + \frac{z^2}{8c^2} + h.o.t. \right]$$

Therefore $k_1(p_0) = \sqrt{a}/b$, $k_2(p_0) = \sqrt{a}/c$, $A = 3\sqrt{a}/b^2$ (A is the coefficient of y^4) and $((A - 3k_1^3)(k_2 - k_1) = -3a(a - b)(b - c)/(b^4c) < 0$. Therefore the ellipse E_{xz} is beak to beak.

Also, let $E = 3\sqrt{a}/c^2$ (coefficient of z^4). So, $(E - 3k_2^3)(k_1 - k_2) = 3a(a - c)(b - c)/(bc^4) > 0$. Therefore, by Theorem 22 the ellipse E_{xy} is zigzag.

Now consider the point $q_0 = (0, -\sqrt{b}, 0)$. The ellipsoid is parametrized by:

$$y(x, z) = -\sqrt{b} + \sqrt{b}\left[\frac{x^2}{2a} + \frac{z^2}{2c} + \frac{x^4}{8a^2} + \frac{x^2z^2}{4ac} + \frac{z^2}{8c^2} + h.o.t.\right]$$

Now, with A coefficient of x^4 and E coefficient of z^4 it follows that $(A - 3k_1^3)(k_2 - k_1) = 3b(a - b)(a - c)/(a^4c) > 0$ and $(E - 3k_2^3)(k_1 - k_2) = 3b(b - c)(a - c)/(ac^4) > 0$.

So both ellipses E_{yz} and E_{xy} are both zigzag, one for the corresponding principal curvature.

b) Hyperboloid of one sheet: the ridges are given by the intersection of the hyperboloid with the coordinates planes.

Consider the point $q_0 = (0, -\sqrt{b}, 0)$. The hyperboloid is parametrized by:

$$y(x, z) = -\sqrt{b} + \sqrt{b}\left[\frac{x^2}{2a} + \frac{z^2}{2c} + \frac{x^4}{8a^2} + \frac{x^2z^2}{4ac} + \frac{z^2}{8c^2} + h.o.t.\right]$$

Therefore $k_1(q_0) = \sqrt{b}/a > 0$, $k_2(q_0) = \sqrt{b}/c < 0$, $A = 3\sqrt{b}/a^2$ (A is the coefficient of x^4) and $E = 3\sqrt{b}/c^2$ coefficient of z^4 it follows that $(A - 3k_1^3)(k_1 - k_2) = -3b(a - b)(a - c)/(a^4c) > 0$ and $(E - 3k_2^3)(k_2 - k_1) = -3b(b - c)(a - c)/(ac^4) < 0$. The hyperbole H_{yz} is beak to beak and the ellipse E_{xy} is zigzag.

Next consider the point $p_0 = (-\sqrt{a}, 0, 0)$. The hyperboloid is parametrized by:

$$x(y, z) = -\sqrt{a} + \sqrt{a}\left[\frac{y^2}{2b} + \frac{z^2}{2c} + \frac{y^4}{8b^2} + \frac{y^2z^2}{4bc} + \frac{z^2}{8c^2} + h.o.t.\right]$$

Therefore $k_1(p_0) = \sqrt{a}/b > 0$, $k_2(p_0) = \sqrt{a}/c < 0$, $A = 3\sqrt{a}/b^2$ (A is the coefficient of y^4) $E = 3\sqrt{a}/c^2$ (coefficient of z^4) it follows that $((A - 3k_1^3)(k_1 - k_2) = 3a(a - b)(b - c)/(b^4c) < 0$ and $(E - 3k_2^3)(k_2 - k_1) = -3a(a - c)(b - c)/(bc^4) < 0$. Therefore the hyperbole H_{xz} and the ellipse E_{xy} are both zigzag.

c) Hyperboloid of two sheets: the ridges are the intersection of the hyperboloid with the coordinates planes.

Consider the point $p_0 = (\sqrt{a}, 0, 0)$. One leaf of the hyperboloid is parametrized by:

$$x(y, z) = \sqrt{a} - \sqrt{a}\left[\frac{y^2}{2b} + \frac{z^2}{2c} + \frac{y^4}{8b^2} + \frac{y^2z^2}{4bc} + \frac{z^2}{8c^2} + h.o.t.\right]$$

Therefore $k_1(p_0) = -\sqrt{a}/b > 0$, $k_2(p_0) = -\sqrt{a}/c > 0$, $A = -3\sqrt{a}/b^2$ (A is the coefficient of y^4) and $((A - 3k_1^3)(k_1 - k_2) = 3a(a - b)(a - c)/(b^4c) < 0$. Therefore the hyperbole H_{xz} is zigzag.

Also, let $E = -3\sqrt{a}/c^2$ (coefficient of z^4). So, $(E - 3k_2^3)(k_2 - k_1) = -3a(a - c)(b - c)/(bc^4) > 0$. Therefore the hyperbole H_{xy} for $|x| < x_0$ is beak to beak.

For $|x| > x_0$ on the hyperbole H_{xy} a similar analysis shows that it is beak to beak. \square

Proposition 34. Consider the ellipsoid $\mathbb{E}_{a,b,c}$ with $a > b > c > 0$.

i) For $c < \lambda < b$ the Darboux curves are and contained in cylindrical region $c < v < \lambda$ and the behavior is as in Fig. 15, upper left.

- ii) For $\lambda = b$ the Darboux curves are the circular sections of the ellipsoid. These circles are contained in planes parallels to the tangent plane to $\mathbb{E}_{a,b,c}$ at the umbilic points. These circles are tangent along the ellipse E_y and through each umbilic point pass only one Darboux curve. See Fig. 15, bottom right.
- iii) For $b < \lambda < a$ the Darboux curves are bounded contained in the two cylindrical region $\lambda \leq u \leq a$ and the behavior is as shown in the Fig. 15, upper right.

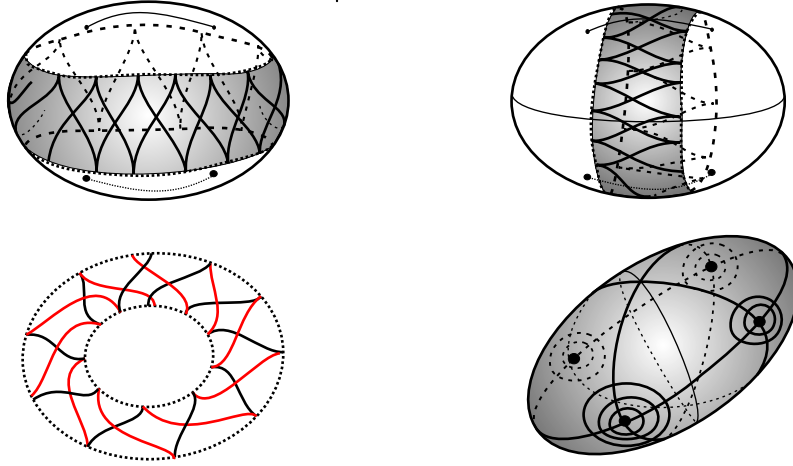


Figure 15: Darboux curves on the ellipsoid.

Proof. First case: $c < \lambda < b$.

The differential equation of Darboux curves is given by:

$$\frac{(v - \lambda)}{H(v)}v'^2 - \frac{(u - \lambda)}{H(u)}u'^2 = 0, \quad c \leq v \leq \lambda \quad \text{and} \quad b \leq u \leq a.$$

Define $d\sigma_1 = \sqrt{(u - \lambda)/H(u)}du$ and $d\sigma_2 = \sqrt{(v - \lambda)/H(v)}dv$.

Therefore the differential equation is equivalent to $d\sigma_1^2 - d\sigma_2^2 = 0$, with $(\sigma_1, \sigma_2) \in [0, L_1] \times [0, L_2]$ ($L_1 = \int_b^a d\sigma_1 < \infty$, $L_2 = \int_c^\lambda d\sigma_2 < \infty$).

In the ellipsoid this analysis implies the following.

The cylindrical region $C_\lambda = \alpha([b, a] \times [c, \lambda])$ is foliated by the integral curves of an implicit differential equation having cusp singularities in ∂C_λ . We observe that this region is free of umbilic point and is bounded by principal curvature lines, in coordinates defined by $v = c$ and $v = \lambda$.

The case $b < \lambda < a$ the analysis is similar. Now the differential equation of Darboux curves are defined in the region $[\lambda, a] \times [c, b]$ and we have a cylindrical region $C_\lambda = \alpha([\lambda, a] \times [c, b])$. This

For $\lambda = b$ the differential equation can be simplified in the following.

$$(u - a)(u - c)dv^2 - (v - a)(v - c)du^2 = 0.$$

This equation is well defined in the rectangle $[c, a] \times [c, a]$.

Define $d\sigma_1 = 1/\sqrt{(u - a)(u - c)}du$ and $d\sigma_2 = 1/\sqrt{(v - a)(v - c)}dv$. So the equation is equivalent $d\sigma_1^2 - d\sigma_2^2 = 0$ with $(\sigma_1, \sigma_2) \in [0, L] \times [0, L]$ ($L = \int_c^a d\sigma_1$). So in this rectangle all

solutions are straight lines. The images of this family of curves on the ellipsoid are its circular sections. In fact we know that the ellipsoid has circular sections parallel to the tangent planes at umbilic points. As the circles are always Darboux lines it follows that the solutions of the differential equation is the family of circular sections. So we have two families of circles having tangency along the ellipse E_y . \square

Proposition 35. *Consider an ellipsoid $\mathbb{E}_{a,b,c}$ with three axes $a > b > c > 0$ and suppose $b < \lambda < a$. Let $L_1 := \int_b^a \sqrt{E(u,b)}du$ and $L_2 := \int_c^\lambda \sqrt{G(b,\lambda)}du$ and define $\rho = \frac{L_2}{L_1}$. Consider the Poincaré map $\pi : \Sigma \rightarrow \Sigma$ associated to the foliation of Darboux curves defined by the implicit differential equation $I = 1/\lambda$.*

Then if $\rho \in \mathbb{R} \setminus \mathbb{Q}$ (resp. $\rho \in \mathbb{Q}$) all orbits are recurrent (resp. periodic) on the cylinder region $v \leq \lambda$. See Figure 15, bottom left.

Proof. The differential equation of Darboux curves is given by:

$$\frac{(v-\lambda)}{H(v)}v'^2 - \frac{(u-\lambda)}{H(u)}u'^2 = 0, \quad c \leq v < \lambda < u \leq a.$$

Define $d\sigma_1 = \sqrt{\frac{(u-\lambda)}{H(u)}}du$ and $d\sigma_2 = \sqrt{\frac{(v-\lambda)}{H(v)}}dv$. By integration, this leads to the chart (σ_1, σ_2) , in a rectangle $[0, L_1] \times [0, L_2]$ in which the differential equation of Darboux is given by

$$d\sigma_1^2 - d\sigma_2^2 = 0.$$

The proof ends with the analysis of the rotation number of the above equation. See similar analysis in Propositions 4 and 35. \square

Proposition 36. *Consider an hyperboloid of one sheet $\mathbb{H}_{a,b,c}$ with $a > b > 0 > c$. Let $\lambda \in (-\infty, c) \cup (b, \infty)$.*

- i) For $\lambda < c$ the Darboux curves are bounded and contained in the cylindrical region $\lambda \leq v \leq c$ and the behavior is as in Fig. 16, upper left.*
- ii) For $b < \lambda < a$ the Darboux curves are unbounded and contained in the cylindrical region $b \leq u \leq \lambda$ (outside the hyperbola E_x) and the behavior is as in Fig. 16, upper right.*
- iii) For $\lambda = c$ and $\lambda = a$ the Darboux curves are straight lines of the hyperboloid. For $\lambda = b$ the solutions are not real. See Fig. 16, bottom left.*
- iv) For $a < \lambda$ all Darboux curves are regular (helices) and goes to ∞ in both directions. See Fig. 16, bottom right.*

Proof. Similar to the proof of Proposition 34. \square

Proposition 37. *Consider a connect component of a hyperboloid of two sheets $\mathbb{H}_{a,b,c}$ with $a > 0 > b > c$.*

- i) For $\lambda < c$ the Darboux curves are non bounded and contained in the non bounded region $v < \lambda$ and the behavior is as in the Fig. 17, left.*

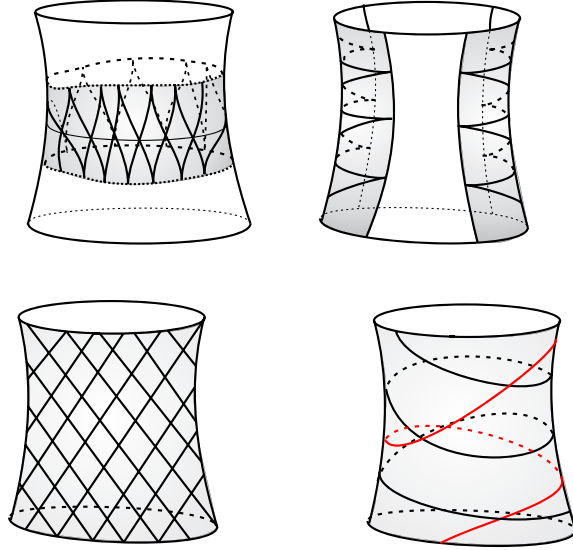


Figure 16: Darboux curves on a hyperboloid of one sheet.

- ii) For $\lambda = c$ the Darboux curves are the circular sections of the hyperboloid. See Fig. 17, center.
- iii) For $c < \lambda < b$ the Darboux curves are non bounded and contained in the cylindrical region $\lambda \leq u \leq b$ and the local behavior is as shown in Fig. 17, right.

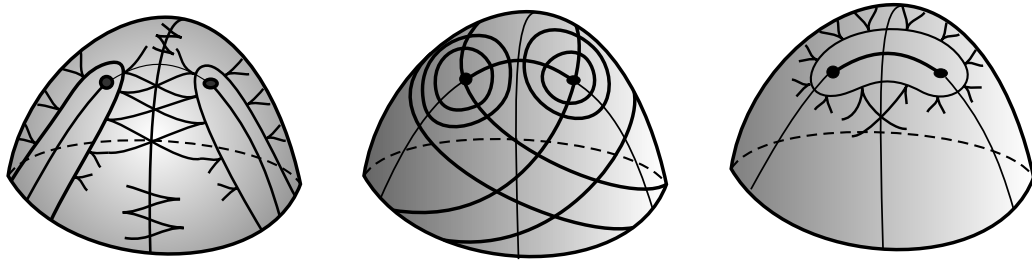


Figure 17: Darboux curves on a connect component of a hyperboloid of two sheets.

Proof. The analysis developed in the case of the ellipsoid also works here. See proof of Proposition 34. □

Remark: The global behavior of geodesics in quadrics, in particular in the ellipsoid, was studied in [Ga-S3].

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