Conformal arc-length via osculating circles

Rémi Langevin and Jun O'Hara*

Institut de Mathématiques de Bourgogne, Université de Bourgogne

Department of Mathematics, Tokyo Metropolitan University

March 7, 2008

Abstract

The set of osculating circles of a given curve in $\boldsymbol{S}^3$ forms a curve in the set of oriented circles in $\boldsymbol{S}^3$. We show that its "$\frac{1}{2}$-dimensional measure" with respect to the pseudo-Riemannian structure of the set of circles is proportional to the conformal arc-length of the original curve, which is a conformally invariant local quantity discovered in the first half of the last century.

Key words and phrases. Conformal arc-length, osculating circles, pseudo-Riemannian manifolds

1991 *Mathematics Subject Classification.* 53A30

1 Introduction

The Frenet-Serret formula provides a local expression of a space curve in terms of the arc-length, the curvature, and the torsion. It is well-known that a space curve is determined by the curvature and the torsion up to motion of $\mathbb{R}^3$, i.e. isometric transformation of $\mathbb{R}^3$.

Let us consider local theory of space curves in conformal geometry. We remark that the arc-length is not preserved by Möbius transformations. Three conformal invariants have been found in search of the normal form. They are conformal arc-length, conformal curvature, and conformal torsion (the reader is referred to [CSW] for example). Just like in the Euclidean case we have:

Theorem 1.1 ([Fi], Theorem 7.2) *An oriented connected vertex-free curve is determined up to conformal motion by the three conformal invariants, the conformal arc-length, the conformal torsion, and the conformal curvature.*

In this article, we study the conformal arc-length.

*This work is supported by the JSPS (Japan Society for the Promotion of Science) Bilateral Program and Grant-in-Aid for Scientific Research No. 19540096.

Definition 1.2 Let C be an oriented curve in $\mathbb{R}^3$. Let s, κ, τ be the arc-length, curvature, and torsion of C respectively. The *conformal arc-length* ρ of C is given by

$$d\rho = \sqrt[4]{\kappa'^2 + \kappa^2\tau^2}\, ds. \tag{1}$$

It gives a conformally invariant parametrization of a vertex-free curve.

It was discovered by [Li] and the above formula was given in [Ta].

In this paper, we give a new interpretation of the conformal arc-length in terms of the set of the osculating circles.

Let $\mathcal{S}(1,3)$ denote the set of the oriented circles in $\mathbb{R}^3$ (or $\boldsymbol{S}^3$), where we consider lines in $\mathbb{R}^3$ as circles. It has a pseudo-Riemannian structure with index 2 which is compatible with the Möbius transformations. Let C be a curve in $\mathbb{R}^3$. The set of the osculating circles to C forms a curve γ in $\mathcal{S}(1,3)$. Our main theorem (Theorem 6.1) claims that the "$L^{\frac{1}{2}}$-measure" of γ

$$L^{\frac{1}{2}}(\gamma) = \lim_{\max|t_{j+1}-t_j|\to+0} \sum_i \sqrt{\|\gamma(t_{i+1}) - \gamma(t_i)\|}$$

is equal to a constant times the conformal arc-length of the original curve C although its length

$$L(\gamma) = \lim_{\max|t_{j+1}-t_j|\to+0} \sum_i \|\gamma(t_{i+1}) - \gamma(t_i)\|$$

does not carry any information as it vanishes for any curve.

This article is arranged as follows. In section 2 we give basics about Minkowski spaces (a good reference about pseudo-Riemannian metrics is [O'N]). In sections 3 and 4 we study codimension 1 cases, namely, a curve in $\mathbb{R}^2$ (or $\boldsymbol{S}^2$) and a curve of the osculating circles which is a lightlike curve in de Sitter space. The section 5 is a preparation to the section 6, where we prove our main theorem. These two sections can be read independent of the previous two sections. In the last section we relate null curves in the space of circles and lightlike curves in the space of spheres and the infinitesimal cross ratio defined in [La-OH] and studied in [O'H2].

In this article we will use the following notations: The letter s is used for the arc-length of a curve in $\mathbb{R}^3$ or other models in the light cone in the Minkowski space, and the derivative with respect to s is denoted by putting $'$. The letter t is for a general parameter, and the derivative with respect to t is denoted by putting $\cdot$. The letter $\tilde{s}$ is for the arc-length of the set of osculating spheres, which is a curve in de Sitter space in the Minkowski space (Section 7).

The authors thank Gil Solanes and Martin Guest for valuable discussions.

2 Preliminaries

Let n be $1, 2$, or 3. The *Minkowski space* $\mathbb{R}^{n+2}_1$ is $\mathbb{R}^{n+2}$ with indefinite inner product:

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + x_1 y_1 + \cdots + x_{n+1} y_{n+1}.$$

Define the *Lorentz form* by $\mathcal{L}(\boldsymbol{v}) = \langle \boldsymbol{v}, \boldsymbol{v} \rangle$. The *norm* of a vector $\boldsymbol{v}$ is given by $\|\boldsymbol{v}\| = \sqrt{|\mathcal{L}(\boldsymbol{v})|}$. A vector $\boldsymbol{v}$ in $\mathbb{R}^{n+2}_1$ is called *spacelike* if $\mathcal{L}(\boldsymbol{v}) > 0$, *lightlike* if $\mathcal{L}(\boldsymbol{v}) = 0$ and $\boldsymbol{v} \neq \boldsymbol{0}$, and *timelike* if $\mathcal{L}(\boldsymbol{v}) < 0$. The set of lightlike vectors and the origin, $\{\boldsymbol{v} \in \mathbb{R}^{n+2}_1 \mid \langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0\}$, is called the *light cone* and shall be denoted by $\boldsymbol{\mathcal{L}}ight$. The "pseudo-sphere", $\{\boldsymbol{v} \in \mathbb{R}^{n+2}_1 | \langle \boldsymbol{v}, \boldsymbol{v} \rangle = 1\}$, is called the *de Sitter space* and shall be denoted by Λ.

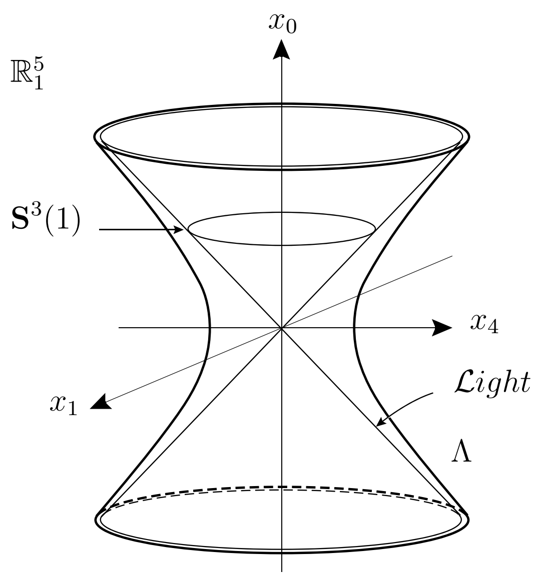

Figure 1: Model of $\mathbf{S}^3$, light cone and de Sitter space

Let W be a vector subspace of $\mathbb{R}^{n+2}_1$. There are three cases which are mutually exclusive. Let $\langle\ ,\ \rangle|_W$ denote the restriction of $\langle\ ,\ \rangle$ to W.

(1) The case when $\langle\ ,\ \rangle|_W$ is non-degenerate. This case can be divided into two cases:

(1-a) The case when $\langle\ ,\ \rangle|_W$ is indefinite. It happens if and only if W intersects the light cone transversely. In this case W is said to be *timelike*.

(1-b) The case when $\langle\ ,\ \rangle|_W$ is positive definite. It happens if and only if W intersects the light cone only at the origin. In this case W is said to be *spacelike*.

(2) The case when $\langle\, , \rangle|_W$ is degenerate. It happens if and only if W is tangent to the light cone. In this case W is said to be *isotropic*.

The sphere $\boldsymbol{S}^n$, the Euclidean space $\mathbb{E}^n$, and the hyperbolic space $\mathbb{H}^n$ can be realized in $\mathbb{R}^{n+2}_1$ as affine sections of the light cone, i.e. the intersection of an affine $(n+1)$-space H and the light cone. Their metrics are induced from the Lorentz form on $\mathbb{R}^{n+2}_1$.

(1) When the affine space H is tangent to the hyperboloid $F_+ = \{\boldsymbol{x} \,|\, \mathcal{L}(\boldsymbol{x}) = -1, x_0 > 0\}$, the intersection $\boldsymbol{\mathcal{L}}ight \cap H$ is a sphere $\boldsymbol{S}^n$ with constant curvature 1.

When the tangent point is $(1, 0, \cdots, 0)$, i.e. when $H = \{x_0 = 1\}$ we will denote it by $\boldsymbol{S}^n(1)$. The n-sphere $\boldsymbol{S}^3$ can be identified with the set of lines through the origin in the light cone.

(2) When the affine space H is parallel to an isotropic subspace and does not contain the origin, the intersection $\boldsymbol{\mathcal{L}}ight \cap H$ is an Euclidean space $\mathbb{E}^n$. For example, take two lightlike vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ given by

$$\boldsymbol{n}_1 = (1, 1, 0, \cdots, 0),\ \boldsymbol{n}_2 = (1, -1, 0, \cdots, 0).$$

Put

$$H = \boldsymbol{n}_2 + (Span\langle \boldsymbol{n}_1 \rangle)^{\perp} \quad \text{and} \quad \mathbb{E}^n_0 = \boldsymbol{\mathcal{L}}ight \cap H.$$

Then the intersection can be explicitly expressed as

$$\mathbb{E}^n_0 = \left\{ \left(1 + \frac{\boldsymbol{x} \cdot \boldsymbol{x}}{4}, -1 + \frac{\boldsymbol{x} \cdot \boldsymbol{x}}{4}, \boldsymbol{x} \right) \,\middle|\, \boldsymbol{x} \in \mathbb{R}^n \right\}, \tag{2}$$

where $\cdot$ denotes the standard inner product. There is an isometry obtained by the projection in the direction of $\boldsymbol{n}_1$ from $\mathbb{E}^n_0$ to $\{(1, -1, \boldsymbol{x}) \,|\, \boldsymbol{x} \in \mathbb{R}^n\} \cong \mathbb{R}^n$.

The lightlike lines in the light cone gives the bijection between $\boldsymbol{S}^n(1) \setminus \{\boldsymbol{n}_1\}$ and $\mathbb{E}^n_0$. It is exactly same as the stereographic projection from the north pole N from $\boldsymbol{S}^n \setminus \{N\}$ to $\mathbb{R}^n$ which is tangent to $\boldsymbol{S}^n$ at the south pole through the identifications $\boldsymbol{S}^n \cong \boldsymbol{S}^n(1)$ and $\mathbb{R}^n \cong \mathbb{E}^n_0$.

3 How to express codimension 1 spheres

3.1 de Sitter space as the space of codimension 1 spheres

Let $\mathcal{S}(n-1, n)$ be the set of oriented $(n-1)$-spheres Σ in $\mathbb{R}^n$ (or $\boldsymbol{S}^n$). Then there is a bijection between $\mathcal{S}(n-1, n)$ and the de Sitter space Λ^{n+1}. Let Σ be an oriented $(n-1)$-sphere in $\mathbb{R}^n$ (or $\boldsymbol{S}^n$). In the Minkowski space $\mathbb{R}^{n+2}_1$, Σ can be realized as the intersecton of the light cone and an oriented $(n+1)$-dimensional subspace Π through the origin (Figure 3). Let $\sigma \in \Lambda^{n+1}$ be the endpoint of the positive unit normal vector to Π. Then the map $\varphi : \mathcal{S}(n-1, n) \ni \Sigma \mapsto \sigma \in \Lambda^{n+1}$ is the bijection we want. Moreover, this bijection is compatible with the action of the Lorentz group $O(4, 1)$, i.e. $\varphi(A \cdot \Sigma) = A\varphi(\Sigma)$ for $A \in O(4, 1)$.

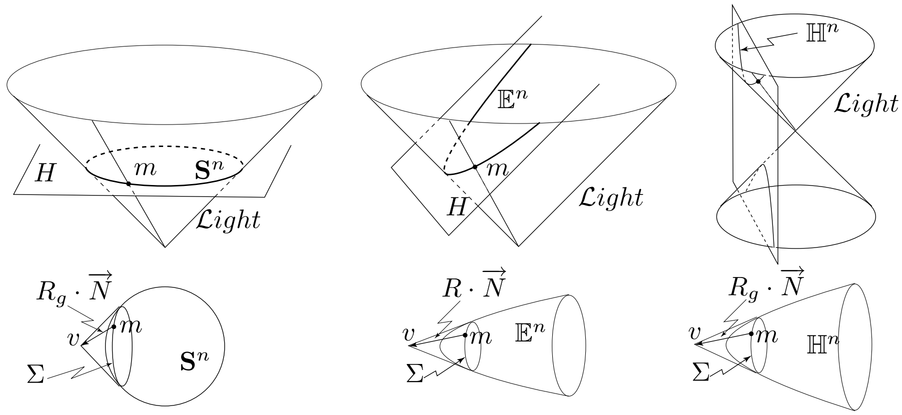

Figure 2: The geodesic radius of curvature

3.2 A point in Λ corresponding to a codimension 1 sphere

Let Σ be a codimension 1 sphere in $\boldsymbol{S}^n$ or in $\mathbb{E}^n$ and m a point in Σ. We will express a point in Λ^{n+1} which corresponds to Σ in terms of the point m, the geodesic curvature k_g of Σ, and the unit normal vector n to Σ at m, where $n \in H$ as the tangent spaces $T_m\boldsymbol{S}^n$ or $T_m\mathbb{E}^n$ are considered as affine subspaces of H (see Figure 2).

Proposition 3.1 *The point $\sigma \in \Lambda$ corresponding to the sphere Σ is*

$$\sigma = k_g m + n$$

Remark: If the orientation of Σ is reversed then the corresponding point σ in Λ should be replaced by $-\sigma$. Therefore, we need a sign convention for the geodesic curvature k_g, which we shall fix as follows.

Choose the unit normal vector n so that if a basis of $T_m\Sigma$ consisting of ordered vectors $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_{n-1}$ gives the positive orientation of $T_m\Sigma$ then a basis of $T_m\boldsymbol{S}^n$ consisting of ordered vectors $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_{n-1}, n$ gives the positive orietation of $T_m\boldsymbol{S}^n$. Let W be an affine n-space in H so that $\Sigma = W \cap \boldsymbol{S}^n$, $\boldsymbol{n}_a(m)$ a unit "inward" normal vector to Σ in W, and p the orthogonal projection to $T_m\boldsymbol{S}^n$. Then k_g should satisfy $k_g > 0$ (or $k_g < 0$) if $p(\boldsymbol{n}_a(m))$ is a positive (or respectively, negative) multiple of n.
When the ambient space is Euclidean one can drop the letter "g" of k_g.

Proof.

The line orthogonal to the hyperplane $\Pi = \mathrm{span}\langle\Sigma\rangle$ of $\mathbb{R}^{n+2}_1$ (the vector subspace containing Σ) is the intersection of the tangent hyperplanes to the light cone at the points of Σ (Figure 2).

This line intersects the affine hyperplane H which contains Σ at the vertex v of a cone tangent to $H \cap \mathcal{L}ight$ along Σ (see the lower part of Figure 2).

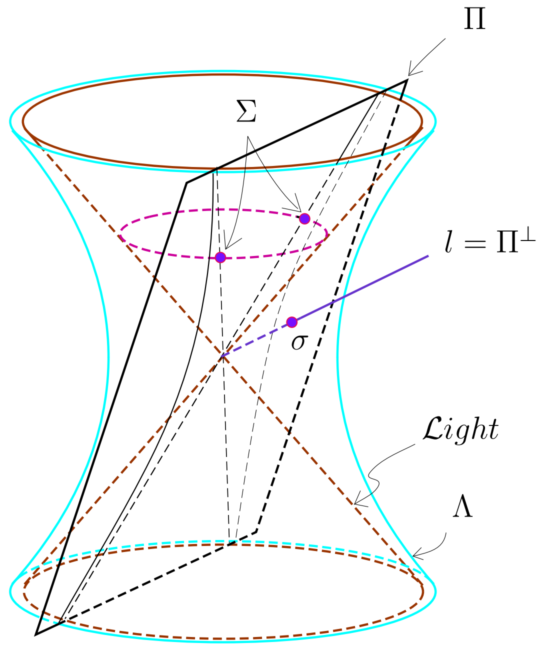

Figure 3: The bijection between $\mathcal{S}(n-1,n)$ and Λ^{n+1}

Notice that the geodesic radius of curvature of a sphere $\Sigma \in \mathbf{S}^n$ or the radius of curvature of a sphere $\Sigma \subset \mathbb{E}^n$ can be seen in a similar way (see Figure 2) as the lengths of a segment from the vertex of the cone v to a point m in Σ where the segment is tangent to $\boldsymbol{S}^n$ or $\mathbb{E}^n$.

In order to see that the length of the vector joining m to v of the cone is equal to the geodesic curvature when Σ is of dimension 1, it is enough to "roll" (which is an isometry) this cone on an Euclidean plane. The curve obtained from Σ is a circle whose curvature is equal to the geodesic curvature of Σ. The radius of the circle is equal to the inverse of the geodesic curvature, which deserves to be called geodesic radius of Σ.

Now σ can be given by $\sigma = c(m + \overrightarrow{mv})$, where c is a constant which makes $\mathcal{L}(\sigma) = 1$. Since m is lightlike, m is orthogonal to $\overrightarrow{mv}$, and $|mv| = \frac{1}{k_g}$ therefore we have $\sigma = k_g(m + \overrightarrow{mv}) = k_g m + n$. □

Notice that the construction would work as well starting from any intersection of the light-cone by any affine space avoiding the origin. The intersection maybe a space of any constant curvature. When the intersection $\mathbb{H}_c = H \cap \boldsymbol{\mathcal{L}}ight$ is hyperbolic the intersection of $\mathbb{H}_c$ with hyperplanes of mixed type gives, on both sheets of $\mathbb{H}_c$, hypersurfaces of constant curvature.

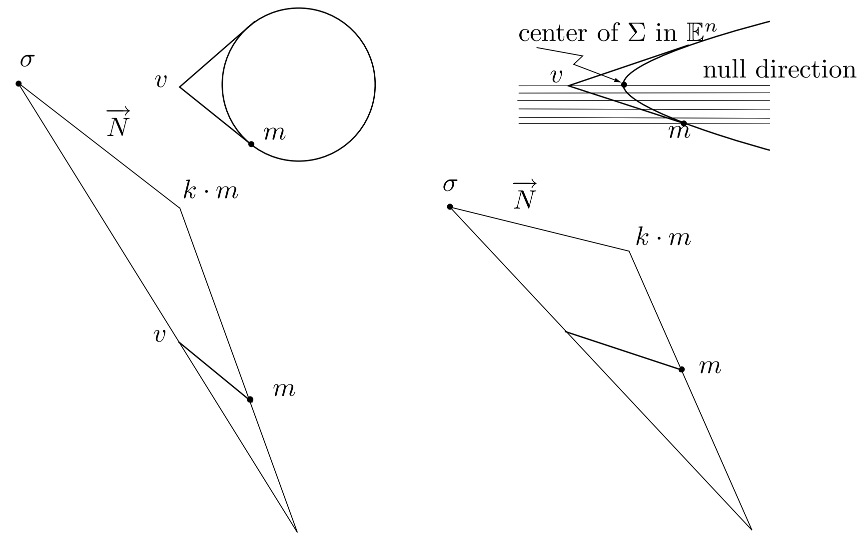

Figure 4: The construction of σ

4 Lightlike curves in de Sitter spaces

In this section we will consider two examples of lightlike curves in Λ: a curve consisting of the osculating circles to a curve in $\mathbb{R}^2$ or $\mathbf{S}^2$, and that of the osculating spheres to a surface in $\mathbb{R}^3$ or $\mathbf{S}^3$ along a corresponding line of principal curvature of the surface.

Proposition 4.1 *The osculating circles along a curve in $\mathbb{R}^2$ or $\mathbf{S}^2$ form a lightlike curve γ in de Sitter space Λ^3.*

Proof. The osculating circle to the curve C at a point m is of geodesic curvature k_g, so the point $\gamma(s)$ which corresponds to it is given by $\gamma(s) = k_g(s)m(s) + n(s)$ by Proposition 3.1.

Let us denote the differential by s by putting $'$. Then we have $\gamma' = {k_g}'m + k_g m' + n'$. By definition of the tangent vector T to C, $m' = T$; as the circle is the osculating circle, its geodesic curvature is k_g, therefore $n' = -k_g T$. So we get $\gamma' = {k_g}'m$, which proves that γ is lightlike. □

4.1 Lightlike curves in $\mathcal{S}(1,2)$

Let us see if the converse is true.

Suppose γ is a lightlike curve in the space $\mathcal{S}(1,2)$ of oriented circles in $\boldsymbol{S}^2$. We identify $\mathcal{S}(1,2)$ with 3-dimensional de Sitter space Λ^3 in $\mathbb{R}^4_1$ as before. Since $\gamma'(t)$ is lightlike, it defines a point in $\boldsymbol{S}^2$ by $\boldsymbol{S}^2 \cap \operatorname{span}\langle\gamma'(t)\rangle$, which shall be denoted by $f(t)$. Put $C = \{f(t)\}$.

Proposition 4.2 *As above, γ is the curve of the osculating circles to C.*

Proof.

Let us denote the differential by t by putting $\bullet$ above. We may assume, after reparametrization if necessary, that the x_0-coordinate of $\dot{\gamma}(t)$ is always equal to 1. Then the point $f(t)$ in $\boldsymbol{S}^2(1)$ is given by $f(t) = \dot{\gamma}(t)$. As $\langle \gamma(t), \gamma(t) \rangle = 1$ and $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle = 0$ we have

$$\langle \gamma(t), \dot{\gamma}(t) \rangle = \langle \gamma(t), \ddot{\gamma}(t) \rangle = \langle \gamma(t), \dddot{\gamma}(t) \rangle = 0.$$

Since the osculating circle to C at $f(t)$ is given by $\boldsymbol{S}^2 \cap \mathrm{span}\langle \dot{\gamma}(t), \ddot{\gamma}(t), \dddot{\gamma}(t) \rangle$ it corresponds to $\pm\gamma(t)$ in Λ^3. $\square$

Let us now prove that the $L^{\frac{1}{2}}$-measure of the lightlike curve $\gamma \subset \Lambda^3$ is equal to the conformal arc-length of C. The norm of a vector $v \in \mathbb{R}^q_p$ is given by

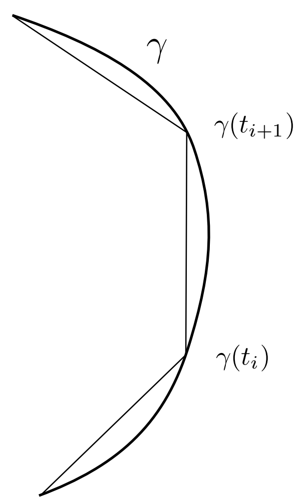

Figure 5: The $\frac{1}{2}$-dimension length of a lightlike curve

$\|v\| = \sqrt{|\langle \boldsymbol{v}, \boldsymbol{v} \rangle|}$. We first prove that an $L^{\frac{1}{2}}$-measure makes sense for all piecewise smooth (C^4 should be enough) lightlike curve in a Lorentz vector space.

Proposition 4.3 *Let γ be a piecewise smooth compact lightlike curve of Λ parametrized by t ($t \in [0,T]$), which is not necessarily the arc-length. We consider a subdivision $0 = t_0 < t_1 \cdots < t_n = T$ of the interval $[0,T]$. Then the following limit exists, and is finite and generically non-zero:*

$$\lim_{\delta \to 0} \sum_i \sqrt{\|\gamma(t_{i+1}) - \gamma(t_i)\|},$$

where $\delta = \sup\{|t_{i+1} - t_i|\}$.

Proof. The tangent vector $\dot{\gamma}$ to γ is lightlike, therefore of zero norm. Derivating $\mathcal{L}(\dot{\gamma})$ we see that the second derivative $\ddot{\gamma}$ is orthogonal to $\dot{\gamma}$. The first non-zero term of a Taylor expansion of $\mathcal{L}(\gamma(t+h)-\gamma(t))$ is of the forth order, and to compute it we need the first three derivatives of γ.

We have

$$\gamma(t+h)-\gamma(t) = h\dot{\gamma} + \frac{h^2}{2}\ddot{\gamma} + \frac{h^3}{3!}\dddot{\gamma} + O(h^4),$$

and hence

$$\mathcal{L}(\gamma(t+h)-\gamma(t)) = \frac{h^4}{4}\ddot{\gamma} + \frac{h^4}{3}\langle\dot{\gamma}, \dddot{\gamma}\rangle + o(h^4).$$

Derivating $0 = \langle\dot{\gamma}, \ddot{\gamma}\rangle$ we get $\langle\dot{\gamma}, \dddot{\gamma}\rangle = -\mathcal{L}(\ddot{\gamma})$, which implies

$$\mathcal{L}(\gamma(t+h)-\gamma(t)) = -\frac{h^4}{12}\mathcal{L}(\ddot{\gamma}) + o(h^4). \tag{3}$$

The norm $\|\gamma(t+h)-\gamma(t)\|$ is therefore of the order of h^2 if the term $\frac{1}{12}\mathcal{L}(\ddot{\gamma})$ is not zero. We will see below a few particular cases of curves where this term is non-zero.

In all cases we have:

$$\lim_{\delta\to 0}\sum_i \sqrt{\|\gamma(t_{i+1})-\gamma(t_i)\|} = \sqrt[4]{\frac{1}{12}}\int_C \sqrt[4]{|\mathcal{L}(\ddot{\gamma})|}\,dt.$$

□

Let s be the arc-length of a curve C.

Theorem 4.4 *The conformal length of a curve C in $\mathbf{S}^2$, $\mathbb{R}^2$, or $\mathbb{H}^2$ is equal to $\sqrt[4]{12}$ times the $\frac{1}{2}$-dimensional measure $L^{\frac{1}{2}}(\gamma)$ of the lightlike curve $\gamma \subset \Lambda^3$ which consists of the osculating circles to C. It is given by $\int_C \sqrt{|k_g'|}ds$ (we can drop the letter "g" when C is a plane curve).*

Proof. Let us denote the differential by s by putting $'$ as before. Proposition 3.1 and the proof of Proposition 4.1 imply that $\gamma = k_g m + n$ and $\gamma' = {k_g}'m$. Therefore the second derivative γ'' is given by ${k_g}''m + {k_g}'T$, where $T = m'$ is the unit tangent vector to C. As the unit vector T is orthogonal to the lightlike vector m and spacelike since all the tangent vectors to the light cone which are not tangent to a generatrix are spacelike, we see that

$$\mathcal{L}(\gamma'') = \mathcal{L}({k_g}''m + {k_g}'T) = {k_g}'^2\mathcal{L}(T) = {k_g}'^2.$$

Therefore, when ${k_g}' \neq 0$, the vector $\gamma(t+h)-\gamma(t)$ is timelike for enough small h. The $L^{\frac{1}{2}}$-measure of γ is

$$L^{\frac{1}{2}}(\gamma) = \sqrt[4]{\frac{1}{12}}\int_C \sqrt{|{k_g}'|}ds.$$

□

4.2 Lightlike curves in $\mathcal{S}(2,3)$

Suppose σ is a lightlike curve in the space $\mathcal{S}(2,3)$ of oriented spheres in $\boldsymbol{S}^3$. We identify $\mathcal{S}(2,3)$ with 3-dimensional de Sitter space Λ in $\mathbb{R}^5_1$ as before.

Since $\sigma'(t)$ is lightlike, it defines a point in $\boldsymbol{S}^3$ by $\boldsymbol{S}^3 \cap \operatorname{span}\langle\sigma'(t)\rangle$, which shall be denoted by $f(t)$. Put $C = \{f(t)\}$.

Proposition 4.5 *As above, a sphere $\Sigma(t)$ which corresponds to $\sigma(t)$ is not necessarily an osculating sphere of C, but it contains the osculating circle to C at $f(t)$.*

Proof. The proof is parallel to that of Proposition 4.2. As $\langle\sigma(t),\sigma(t)\rangle = 1$ and $\langle\dot\sigma(t),\dot\sigma(t)\rangle = 0$ we have

$$\langle\sigma(t),\dot\sigma(t)\rangle = \langle\sigma(t),\ddot\sigma(t)\rangle = \langle\sigma(t),\dddot\sigma(t)\rangle = 0.$$

Therefore $\operatorname{span}\langle\sigma(t)\rangle$ is contained by $\left(\operatorname{span}\langle\dot\sigma(t),\ \ddot\sigma(t),\ \dddot\sigma(t)\rangle\right)^{\perp}$, which implies $\Sigma(t) = \boldsymbol{S}^3 \cap (\operatorname{span}\langle\sigma(t)\rangle)^{\perp}$ contains $\boldsymbol{S}^3 \cap \operatorname{span}\langle\dot\sigma(t),\ \ddot\sigma(t),\ \dddot\sigma(t)\rangle$ which is the osculating circle of C at $f(t)$. □

In general, Proposition 7.8 implies the following:

Corollary 4.6 *Let $\bar C = \{\bar m(t)\}$ be a curve in $\boldsymbol{S}^3$. Let Σ_0 be a sphere containing the osculating circle to $\bar C$ at $\bar m(t_0)$. Then there is the unique lightlike curve $\bar\sigma$ in $\mathcal{S}(2,3)$ that satisfies the following conditions:*

(1) *$\bar\sigma(t_0)$ corresponds to Σ_0.*

(2) *The sphere corresponding to $\bar\sigma(t)$ contains the osculating circle to the curve $\bar C$ at $\bar m(t)$.*

This is because a sphere has second order contact with a curve if and only if it contains an osculating circle of the curve.

Let us now consider a surface M in $\mathbf{S}^3$, $\mathbb{H}^3$, or $\mathbb{R}^3$. The geodesic curvature of a sphere Σ contained in $\mathbf{S}^3$ or $\mathbb{H}^3$ means the geodesic curvature of any geodesic curve on it. In the Poincaré ball model, the intersection of a 2-sphere with the ball is called a sphere. It may be a usual geodesic sphere or a plane of constant curvature between 0 and -1.

At a point $m \in M$, a sphere tangent to M at m whose geodesic curvature is equal to one of the principal curvatures k_1, k_2 of M at m, has higher contact with M. Let us call Σ_1 and Σ_2 the two *osculating spheres* to M at m. They are distinct if the point m is not an umbilical point of M.

Theorem 4.7 *The curve $\gamma_1 \subset \Lambda^4$ corresponding to the osculating spheres Σ_1 along a line of principal curvature C_1 for the principal curvature k_1 is lightlike. Its $\frac{1}{2}$-dimensional measure is*

$$L^{\frac{1}{2}}(\gamma_1) = \sqrt[4]{\frac{1}{12}} \int_{C_1} \sqrt{|X_1(k_1)|}\, ds$$

where X_1 is the unit tangent vector to C_1 which is parametrized by its arc-length s.

Proof. The proof is the same as above (Theorem 4.1) using the formula $\gamma(s) = k_g m(s) + n(s)$. The hypothesis that C_1 is a line of principal curvature associated to the principal curvature k_1 implies that $n'(s) = k_g X_1(s)$. $\square$

Remark: A similar statement is also valid if M is an hypersurface of some space-form. The integral $\int_{C_1} \sqrt{|X_1(k_1)|}ds$ has already appeared in [Ro-Sa].

5 Space $\mathcal{S}(1,3)$ of the oriented circles in $\boldsymbol{S}^3$

The pseudo-Riemannian structure of the space of the oriented circles in $\mathbb{R}^3$ (or $\boldsymbol{S}^3$) also arises naturally in the study of conformal geometry. Each tangent space of $\mathcal{S}(1,3)$ has an indefinite non-degenerate quadratic form which is compatible with Möbius transformations. The reader is referred to [La-OH2] for the details.

5.1 The set of circles as a Grassmann manifold

An oriented circle in $\mathbb{R}^3$ (or $\boldsymbol{S}^3$) can be realized in the Minkowski space $\mathbb{R}^5_1$ as the intersection of the light cone and an oriented timelike 3-dimensional vector subspace. Therefore the set $\mathcal{S}(1,3)$ can be identified with the Grassmann manifold $\widetilde{Gr}_-(3;\mathbb{R}^5_1)$ of oriented 3-dimensional timelike subspaces of $\mathbb{R}^5_1$. It follows that the set $\mathcal{S}(1,3)$ is a homogeneous space $SO(4,1)/SO(2)\times SO(2,1)$. We give its pseudo-Riemannian structure explicitly in what follows.

5.2 Plücker coordinates for the set of circles

Let us recall the Plücker coordinates of Grassmannian manifolds.

Let W be an oriented 3-dimensional vector subspace in $\mathbb{R}^5_1$, and let $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\}$ be an ordered basis of W which gives the orientation of W. Define $p_{i_1 i_2 i_3}$ $(0 \le i_k \le 4)$ by

$$p_{i_1 i_2 i_3} = \begin{vmatrix} x_{1\,i_1} & x_{1\,i_2} & x_{1\,i_3} \\ x_{2\,i_1} & x_{2\,i_2} & x_{2\,i_3} \\ x_{3\,i_1} & x_{3\,i_2} & x_{3\,i_3} \end{vmatrix}. \tag{4}$$

They satisfy the *Plücker relations*:

$$\sum_{k=1}^{4} (-1)^k p_{i_1 i_2 j_k} p_{j_1 \cdots \widehat{j_k} \cdots j_4} = 0, \tag{5}$$

where $\widehat{j_k}$ indicates that the index j_k is being removed.

As we are concerned with the orientation of the subspaces, we use the Euclidean spaces for the Plücker coordinates in this article instead of the projective spaces which are used in most cases. The *exterior product* of $\boldsymbol{x}_1, \boldsymbol{x}_2$, and $\boldsymbol{x}_3$ in $\mathbb{R}^5_1$ is given by

$$\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3 = (\cdots, p_{i_1 i_2 i_3}, \cdots) \in \mathbb{R}^{10} \qquad (i_1 < i_2 < i_3)$$

through the identification $\bigwedge^3 \mathbb{R}^5 \cong \mathbb{R}^{10}$.

5.3 Pseudo-Riemannian structure of $\bigwedge^3 \mathbb{R}^5_1$

The indefinite inner product of the Minkowski space $\mathbb{R}^5_1$ naturally induces that of $\bigwedge^3 \mathbb{R}^5_1$ by

$$\langle \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3, \boldsymbol{y}_1 \wedge \boldsymbol{y}_2 \wedge \boldsymbol{y}_3 \rangle = - \begin{vmatrix} \langle \boldsymbol{x}_1, \boldsymbol{y}_1 \rangle & \langle \boldsymbol{x}_1, \boldsymbol{y}_2 \rangle & \langle \boldsymbol{x}_1, \boldsymbol{y}_3 \rangle \\ \langle \boldsymbol{x}_2, \boldsymbol{y}_1 \rangle & \langle \boldsymbol{x}_2, \boldsymbol{y}_2 \rangle & \langle \boldsymbol{x}_2, \boldsymbol{y}_3 \rangle \\ \langle \boldsymbol{x}_3, \boldsymbol{y}_1 \rangle & \langle \boldsymbol{x}_3, \boldsymbol{y}_2 \rangle & \langle \boldsymbol{x}_3, \boldsymbol{y}_3 \rangle \end{vmatrix} \tag{6}$$

for any $\boldsymbol{x}_i$ and $\boldsymbol{y}_j$ in $\mathbb{R}^5_1$.

The above formula is a natural generalization of the one for the exterior products of four vectors which we obtained in [La-OH], where we studied the set of oriented spheres in $\boldsymbol{S}^3$.

It follows that $\bigwedge^3 \mathbb{R}^5_1$ can be identified with $\mathbb{R}^{10}$ with the pseudo-Riemannian structure with index 4, which we denote by $\mathbb{R}^{10}_4$, so that $\{\boldsymbol{e}_{i_1} \wedge \boldsymbol{e}_{i_2} \wedge \boldsymbol{e}_{i_3}\}_{i_1<i_2<i_3}$ is a pseudo-orthonormal basis of $\bigwedge^3 \mathbb{R}^5_1$ with

$$\langle \boldsymbol{e}_{i_1} \wedge \boldsymbol{e}_{i_2} \wedge \boldsymbol{e}_{i_3}, \, \boldsymbol{e}_{i_1} \wedge \boldsymbol{e}_{i_2} \wedge \boldsymbol{e}_{i_3} \rangle = \begin{cases} -1 & \text{if} \quad i_1 \geq 1, \\ +1 & \text{if} \quad i_1 = 0. \end{cases} \tag{7}$$

Let us realize $\mathcal{S}(1,3)$ as a pseudo-Riemannian submanifold of $\bigwedge^3 \mathbb{R}^5_1 \cong \mathbb{R}^{10}_4$.

Lemma 5.1 *Let W be an oriented 3-dimensional vector subspace in $\mathbb{R}^5_1$ spanned by $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$. Then W is timelike if and only if*

$$\langle \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3, \, \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3 \rangle > 0,$$

and isotropic (i.e. tangent to the light cone) if and only if

$$\langle \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3, \, \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3 \rangle = 0.$$

Proof. Case (1). Suppose W is not isotropic. We may assume without loss of generality that $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\}$ is a pseudo-orthonormal basis of W. If W is timelike, one of $\boldsymbol{x}_1, \boldsymbol{x}_2$ and $\boldsymbol{x}_3$ is timelike, and therefore $\langle \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3, \, \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3 \rangle = 1$ by (6). If W is spacelike, then $\langle \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3, \, \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3 \rangle = -1$.

Case (2). Suppose W is isotropic. Then W is tangent to the light cone at a lightlike line l. Now we may assume without loss of generality that $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\}$

is a pseudo-orthogonal basis of W and that $\boldsymbol{x}_1$ belongs to l. Then we have $\langle \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3, \, \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3 \rangle = 0$. □

This Lemma and Lemma 5.6 in the next subsubsection imply

Proposition 5.2 *Let $\Theta(1,3)$ be the intersection of the quadric satisfying the Plücker relations and the unit pseudo-sphere:*

$$\Theta(1,3) = \left\{ (\cdots, p_{i_1 i_2 i_3}, \cdots) \in \mathbb{R}^{10}_4 \;\middle|\; \begin{array}{l} \displaystyle\sum_{k=1}^{4} (-1)^k p_{i_1 i_2 j_k} p_{j_1 \cdots \widehat{j_k} \cdots j_4} = 0 \\ \displaystyle -\sum_{i_1 \geq 1} {p_{i_1 i_2 i_3}}^2 + \sum_{i_2 \geq 1} {p_{0 i_2 i_3}}^2 = 1 \end{array} \right\}. \quad (8)$$

It is a 6-dimensional pseudo-Riemannian submanifold of $\mathbb{R}^{10}_4$ with pseudo-Riemannian structure of index 2.

Then the set $\mathcal{S}(1,3)$ of oriented circles in $\boldsymbol{S}^3$ can be identified with with the submanifold $\Theta(1,3)$ in $\mathbb{R}^{10}_4$ through a bijection ψ given by

$$\begin{array}{ccccccc} \psi: \ \mathcal{S}(1,3) & \stackrel{\cong}{\longrightarrow} & \widetilde{Gr}_-(3;\mathbb{R}^5_1) & \stackrel{\cong}{\longrightarrow} & \Theta(1,3) & \subset \bigwedge^3 \mathbb{R}^5_1 \cong \mathbb{R}^{10}_4 \\ \cup & & \cup & & \cup & \\ W \cap \boldsymbol{S}^3(\infty) & \mapsto & W = Span\langle \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \rangle & \mapsto & \dfrac{\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3}{\|\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3\|}. & \end{array} \quad (9)$$

5.4 Conformal invariance of the pseudo-Riemannian structure

We show that the pseudo-Riemannian structure of $\mathcal{S}(1,3)$ is conformally invariant, namely, a Möbius transformation of $\boldsymbol{S}^3$ induces a pseudo-orthogonal transformation of $\Theta(1,3) \subset \bigwedge^3 \mathbb{R}^5_1$. Let $O(6,4)$ denote the pseudoorthogonal group.

Definition 5.3 Define a map $\Psi : M_5(\mathbb{R}) \to M_{10}(\mathbb{R})$ by

$$\Psi : M_5(\mathbb{R}) \ni A = (a_{ij}) \mapsto \Psi(A) = (\tilde{a}_{IJ}) \in M_{10}(\mathbb{R}),$$

where $I = (i_1 i_2 i_3)$ and $J = (j_1 j_2 j_3)$ are multi-indices, and $\tilde{a}_{IJ}$ is given by

$$\tilde{a}_{IJ} = \begin{vmatrix} a_{i_1 j_1} & a_{i_1 j_2} & a_{i_1 j_3} \\ a_{i_2 j_1} & a_{i_2 j_2} & a_{i_2 j_3} \\ a_{i_3 j_1} & a_{i_3 j_2} & a_{i_3 j_3} \end{vmatrix}.$$

Lemma 5.4 (1) *We have*

$$(A\boldsymbol{x}_1) \wedge (A\boldsymbol{x}_2) \wedge (A\boldsymbol{x}_3) = \Psi(A)\,(\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3) \qquad (\forall \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \in \mathbb{R}^5_1) \quad (10)$$

for $A \in M_5(\mathbb{R})$.

(2) *If* $A \in O(4,1)$ *then* $\Psi(A) \in O(6,4)$.

(3) *The restriction of* Ψ *to* $O(4,1)$ *is a homomorphism.*

We can say more, although we do not give proof: The matrix $\Psi(A)$ can be characterized by (10). The reverse statement of (2) also holds. The restriction of Ψ to $Gl(5,\mathbb{R})$ is a homomorphism whose kernel consists of $\{\pm I\}$.

Proof. (1) The definition of $\tilde{a}_{IJ}$ implies

$$\Psi(A)\,(\boldsymbol{e}_{i_1} \wedge \boldsymbol{e}_{i_2} \wedge \boldsymbol{e}_{i_3}) = (A\boldsymbol{e}_{i_1}) \wedge (A\boldsymbol{e}_{i_2}) \wedge (A\boldsymbol{e}_{i_3}).$$

(2) If $A \in O(4,1)$ then (6) and (10) imply

$$\langle \Psi(A)(\boldsymbol{e}_{i_1} \wedge \boldsymbol{e}_{i_2} \wedge \boldsymbol{e}_{i_3}), \Psi(A)(\boldsymbol{e}_{j_1} \wedge \boldsymbol{e}_{j_2} \wedge \boldsymbol{e}_{j_3})\rangle = \langle \boldsymbol{e}_{i_1} \wedge \boldsymbol{e}_{i_2} \wedge \boldsymbol{e}_{i_3}, \boldsymbol{e}_{j_1} \wedge \boldsymbol{e}_{j_2} \wedge \boldsymbol{e}_{j_3}\rangle,$$

which implies $\Psi(A) \in O(6,4)$.

(3) Routine calculation in linear algebra implies $\Psi(AB)_{IJ} = \sum_K \tilde{a}_{IK}\tilde{b}_{KJ}$. □

Corollary 5.5 *We have*

$$\psi(A \cdot \Gamma) = \Psi(A)\psi(\Gamma)$$

for $\Gamma \in \mathcal{S}(1,3)$ *and* $A \in O(4,1)$, *where* ψ *is the bijection from* $\mathcal{S}(1,3)$ *to* $\Theta(1,3) \subset \mathbb{R}^{10}_4$ *given by* (9) *and* Ψ *the homomorphism from* $O(4,1)$ *to* $O(6,4)$ *given in* Definition 5.3.

Lemma 5.6 *The restriction of the indefinite inner product of* $\mathbb{R}^{10}_4$ *to each tangent space of* $\Theta(1,3)$ *induces a non-degenerate quadratic form of index* 2.

Proof. The conformal invariance of the pseudo-Riemannian structure allows us to assume that an oriented circle Γ passes through $(\pm 1,0,0,0)$ and $(0,1,0,0)$. The index can be calculated in several ways.

(i) Γ corresponds to $W = Span\langle \boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2\rangle$ in the Grassmannian $\widetilde{Gr}_-(3;\mathbb{R}^5_1)$. The tangent space $T_W\widetilde{Gr}_-(2;\mathbb{R}^5_1)$ is isomorphic to $\mathrm{Hom}\left(W, W^\perp\right)$, which is isomorphic to $M_{3,2}(\mathbb{R})$. We can construct six vectors which form a pseudo-orthonormal basis of the tangent space explicitly. It turns out that two of them are timelike and the other four are spacelike.

(ii) The tangent space $T_\Gamma\Theta(1,3)$ can be identified with the pseudo-orthogonal complements of the subspace spanned by gradients of the defining functions of $\Theta(1,3)$ which appear in (8). There are five non-trivial Plücker relations and exactly three of them are independent. Two of them give timelike gradients and the rest gives a spacelike one. On the other hand, the gradient of $-\sum_{i_1\geq 1} {p_{i_1i_2}}^2 + \sum_{i_2\geq 1} {p_{0i_2}}^2 - 1$ is spacelike. Hence the index can be given by $4-2=2$. □

The index can also be implied by Proposition 3.2.6 of [Ko-Yo] as $\mathcal{S}(1,3)$ is a homogeneous space $SO(4,1)/SO(2) \times SO(2,1)$.

6 Osculating circles and the conformal arc-length

Let us realize $\mathbb{R}^3$ in the Minkowski space $\mathbb{R}^5_1$ as the isotropic affine section of the light cone $\boldsymbol{\mathcal{L}}\mathit{ight}$ given by (2) in section 2. We use the following notation in what follows. Let $C = \{m(s)\}$ be an oriented curve in $\mathbb{R}^3$ parametrized by the arc-length s. Let $\bar{m}$ be a map which is induced from m;

$$\bar{m}(s) = \left(1 + \frac{m(s)\cdot m(s)}{4},\ -1 + \frac{m(s)\cdot m(s)}{4},\ m(s)\right) \in \mathbb{E}^3_0 \subset \mathbb{R}^5_1,$$

where $\mathbb{E}^3_0$ is given by formula (2).

An *osculating circle* of a curve C at a point x is the circle with the best contact with C at x. We will denote it by $\mathcal{O}_x$. It has the contact of second order with C at x.

Suppose x, y, and z are points on C. When x, y, and z are mutually distinct, let $\Gamma(x, y, z)$ denote the circle that passes through the points x, y, and z in K whose orientation is given by the cyclic order of $\{x, y, z\}$. When two (or three) of the points x, y, and z coincide, $\Gamma(x, y, z)$ means a tangent circle (or respectively, an osculating circle) whose orientation coincides with that of C at the tangent point.

Let $\gamma(u, v, w)$ be a point in $\Theta(1, 3) \subset \mathbb{R}^{10}_4$ which corresponds to a circle $\Gamma(\bar{m}(u), \bar{m}(v), \bar{m}(w))$ through the bijection ψ from $\mathcal{S}(1, 3)$ to $\Theta(1, 3)$ (see (9)). Put $\gamma(u) = \gamma(u, u, u)$. It corresponds to the osculating circle at $m(u)$.

6.1 Nullity of the curve of the osculating circles

Theorem 6.1 *Let γ be a curve in the set $\mathcal{S}(1, 3)$ of oriented circles which corresponds to the set of the osculating circles of a curve C in $\boldsymbol{S}^3$. Then the curve γ is a null curve, i.e. $\langle \gamma', \gamma' \rangle \equiv 0$.*

Proof.

Let us express the osculating circles using the exterior products of vectors in the Minkowski space.

Observe that $\langle \bar{m}, \bar{m} \rangle \equiv 0$, as $\bar{m}$ belongs to the light cone, and that

$$\langle \bar{m}', \bar{m}' \rangle \equiv m' \cdot m' \equiv 1,$$

where $\cdot$ denotes the standard inner product of $\mathbb{R}^3$. Define F_2 and F_3 by

$$F_2 = \langle \bar{m}'', \bar{m}'' \rangle,\ F_3 = \langle \bar{m}''', \bar{m}''' \rangle. \tag{11}$$

Then they satisfy

$$\begin{aligned} F_2 &= m'' \cdot m'' = \kappa^2, \\ F_3 &= m''' \cdot m''' = \kappa^4 + \kappa'^2 + \kappa^2\tau^2. \end{aligned}$$

By derivating these equations we obtain a table of $\langle \bar{m}^{(i)}(0), \bar{m}^{(j)}(0) \rangle$ needed in this article (Table 1).

	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$
$i=0$	0	0	-1	0	F_2
$i=1$		1	0	$-F_2$	$*$
$i=2$			F_2	$*$	$*$
$i=3$				F_3	$*$

Table 1: A table of $\langle \bar{m}^{(i)}(0), \bar{m}^{(j)}(0)\rangle \quad (F_2 = \kappa^2,\ F_3 = \kappa^4 + \kappa'^2 + \kappa^2\tau^2)$

Put

$$\begin{aligned} D^{l_1 l_2 l_3}_{k_1 k_2 k_3} &= -\langle \bar{m}^{(k_1)} \wedge \bar{m}^{(k_2)} \wedge \bar{m}^{(k_3)}, \bar{m}^{(l_1)} \wedge \bar{m}^{(l_2)} \wedge \bar{m}^{(l_3)}\rangle \\ &= \det\left(\langle \bar{m}^{(k_i)}, \bar{m}^{(l_j)}\rangle\right) \\ &= \begin{vmatrix} \langle \bar{m}^{(k_1)}, \bar{m}^{(l_1)}\rangle & \langle \bar{m}^{(k_1)}, \bar{m}^{(l_2)}\rangle & \langle \bar{m}^{(k_1)}, \bar{m}^{(l_3)}\rangle \\ \langle \bar{m}^{(k_2)}, \bar{m}^{(l_1)}\rangle & \langle \bar{m}^{(k_2)}, \bar{m}^{(l_2)}\rangle & \langle \bar{m}^{(k_2)}, \bar{m}^{(l_3)}\rangle \\ \langle \bar{m}^{(k_3)}, \bar{m}^{(l_1)}\rangle & \langle \bar{m}^{(k_3)}, \bar{m}^{(l_2)}\rangle & \langle \bar{m}^{(k_3)}, \bar{m}^{(l_3)}\rangle \end{vmatrix} \end{aligned} \tag{12}$$

If $u < v < w$ then $\bar{m}(u), \bar{m}(v)$, and $\bar{m}(w)$ are linearly independent in $\mathbb{R}^5_1$, and therefore (9) implies that $\gamma(u,v,w)$ is given by

$$\gamma(u,v,w) = \frac{\bar{m}(u) \wedge \bar{m}(v) \wedge \bar{m}(w)}{\left\|\bar{m}(u) \wedge \bar{m}(v) \wedge \bar{m}(w)\right\|}. \tag{13}$$

Lemma 6.2 *The osculating circles are expressed in* $\Theta(1,3)$ *by*

$$\gamma(s) = \bar{m}(s) \wedge \bar{m}'(s) \wedge \bar{m}''(s). \tag{14}$$

Proof. Let us consider the limit of $\gamma(u,v,w)$ as both u and w approach v. Taylor's expansion formula implies

$$\begin{aligned} &\bar{m}(s - \Delta s) \wedge \bar{m}(s) \wedge \bar{m}(s + \Delta s') \\ &= \left(\bar{m}(s) - \Delta s \bar{m}'(s) + \frac{(\Delta s)^2}{2}\bar{m}''(s) + O\left((\Delta s)^3\right)\right) \wedge \bar{m}(s) \\ &\quad \wedge \left(\bar{m}(s) + \Delta s' \bar{m}'(s) + \frac{(\Delta s')^2}{2}\bar{m}''(s) + O\left((\Delta s')^3\right)\right) \\ &= \frac{\Delta s \Delta s'(\Delta s + \Delta s')}{2}\, \bar{m}(s) \wedge \bar{m}'(s) \wedge \bar{m}''(s) + \text{higher order terms}. \end{aligned}$$

It follows that the osculating circle is given by the vector $\bar{m}(s) \wedge \bar{m}'(s) \wedge \bar{m}''(s)$ multiplied by a positive number. By the formulae (6) and Table 1 we have

$$\left\langle \bar{m} \wedge \bar{m}' \wedge \bar{m}'', \bar{m} \wedge \bar{m}' \wedge \bar{m}''\right\rangle = -D^{012}_{012} = -\begin{vmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & m'' \cdot m'' \end{vmatrix} = 1.$$

Therefore the osculating circle $\gamma(s)$ is given by

$$\gamma(s) = \frac{\bar{m}(s) \wedge \bar{m}'(s) \wedge \bar{m}''(s)}{\left\| \bar{m}(s) \wedge \bar{m}'(s) \wedge \bar{m}''(s) \right\|} = \bar{m}(s) \wedge \bar{m}'(s) \wedge \bar{m}''(s).$$

□

(Proof of Theorem 6.1 continued:)

The formula (14) implies

$$\gamma'(s) = \bar{m}(s) \wedge \bar{m}'(s) \wedge \bar{m}'''(s). \tag{15}$$

By the formulae (6) and Table 1 we have

$$\left\langle \bar{m} \wedge \bar{m}' \wedge \bar{m}''', \bar{m} \wedge \bar{m}' \wedge \bar{m}''' \right\rangle = -D^{013}_{013} = - \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & * \\ 0 & * & * \end{vmatrix} = 0,$$

which implies that $|\gamma'(s)| \equiv 0$. This ends the proof of Theorem 6.1. □

6.2 Conformal arc-length via osculating circles

Theorem 6.3 *Let γ be a curve in the set $\mathcal{S}(1,3)$ of oriented circles which corresponds to the set of the osculating circles of a curve C in $\boldsymbol{S}^3$. The conformal arc-length ρ of the curve C satisfies*

$$\rho = \sqrt[4]{12} \lim_{\max|t_{j+1}-t_j| \to +0} \sum_i \sqrt{\|\gamma(t_{i+1}) - \gamma(t_i)\|}\,. \tag{16}$$

Proof. Let us abbreviate $\gamma^{(i)}(0)$ and $\bar{m}^{(i)}(0)$ as $\gamma^{(i)}$ and $\bar{m}^{(i)}$ in the proof. Since γ is a null curve, just like in the case of a lightlike curve which we saw in equation (3), we have

$$\langle \gamma(s) - \gamma(0), \gamma(s) - \gamma(0) \rangle = -\frac{1}{12} \langle \gamma'', \gamma'' \rangle\, s^4 + O(s^5).$$

Since

$$\gamma'' = \bar{m} \wedge \bar{m}' \wedge \bar{m}^{(4)} + \bar{m} \wedge \bar{m}'' \wedge \bar{m}''',$$

the formula (6) and Table 1 imply

$$\begin{aligned} \langle \gamma'', \gamma'' \rangle &= -\left(D^{014}_{014} + 2D^{023}_{014} + D^{023}_{023} \right) \\ &= -\left(\begin{vmatrix} 0 & 0 & F_2 \\ 0 & 1 & * \\ F_2 & * & * \end{vmatrix} + 2 \begin{vmatrix} 0 & -1 & 0 \\ 0 & 0 & -F_2 \\ F_2 & * & * \end{vmatrix} + \begin{vmatrix} 0 & -1 & 0 \\ -1 & * & * \\ 0 & * & F_3 \end{vmatrix} \right) \\ &= F_3 - F_2^{\,2}\,. \end{aligned} \tag{17}$$

Since $F_3 - F_2^2 = {\kappa'}^2 + \kappa^2\tau^2$ we have

$$\|\gamma(s)-\gamma(0)\|^2 = |\langle \gamma(s)-\gamma(0), \gamma(s)-\gamma(0)\rangle| = \frac{{\kappa'}^2+\kappa^2\tau^2}{12}s^4 + O(s^5), \quad (18)$$

which, compared with (1), completes the proof. □

Remark: A pair of nearby osculating circles is a "*timelike pair*", i.e. $\gamma(s+\Delta s)-\gamma(s)$ is timelike for $|\Delta s| \ll 1$.

Corollary 6.4 *The conformal arc-length ρ satisfies*

$$d\rho = \sqrt[4]{\langle \gamma'', \gamma''\rangle}\, ds, \quad (19)$$

where s is the arc-length of the original curve in $\mathbb{R}^3$, γ is the curve in $\mathcal{S}(1,3)$ consisting of osculating circles, and $\gamma'' = \frac{d^2\gamma}{ds^2}$.

Corollary 6.5 *Let $C = \{m(s)\}$ be a curve in $\mathbb{R}^N$ parametrized by the arc-length s and $'$ denote the differential by s. Then the 1-form ω_C on the curve given by*

$$\omega_C = \sqrt[4]{m'''\cdot m''' - (m''\cdot m'')^2}\, ds$$

is invariant under Möbius transformations. Namely, if G is a Möbius transformation then we have $G^*\omega_{G\cdot C} = \omega_C$. When $N=3$ ω_C is equal to $d\rho$, where ρ is the conformal arc-length of C.

Proof. Let us prove only in the case when $N=3$. Suppose $G\cdot C$ are expressed as $G\cdot C = \{m_2(t)\}$ with t being the arc-length of $G\cdot C$. Let γ_m and γ_{m_2} be the curves in $\mathcal{S}(1,3)$ which consists of the osculating circles to C and $G\cdot C$ respectively. The formula (17) implies that

$$\sqrt[4]{m'''\cdot m''' - (m''\cdot m'')^2}\, ds = \sqrt[4]{\langle \gamma_m'', \gamma_m''\rangle}\, ds,$$

$$\sqrt[4]{\dddot{m}_2\cdot \dddot{m}_2 - (\ddot{m}_2\cdot\ddot{m}_2)^2}\, dt = \sqrt[4]{\left\langle \ddot{\gamma}_{m_2}, \ddot{\gamma}_{m_2}\right\rangle}\, dt,$$

where putting $\cdot$ above means taking the differential by t.

Lemma 5.4 shows that the Möbius transformation $G \in O(4,1)$ produces a pseudoorthogonal transformation $\widetilde{G} \in O(6,4)$, and γ_{m_2} is given by $\gamma_{m_2} = \widetilde{G}\cdot\gamma_m$. Since $\langle \ddot{\gamma}_{m_2}, \ddot{\gamma}_{m_2}\rangle = \langle \widetilde{G}\cdot\ddot{\gamma}_m, \widetilde{G}\cdot\ddot{\gamma}_m\rangle = \langle \ddot{\gamma}_m, \ddot{\gamma}_m\rangle$, Theorem 6.1 (1) implies that the Corollary is a consequence of the following Lemma. □

Corollaries 6.4 and 6.5 shows that the conformal arc-length is in fact invariant under Möbius transformations.

Lemma 6.6 *Let γ be a null (or lightlike) curve. Let t and u be any parameters of γ. Then we have*

$$\left\langle \frac{d^2\gamma}{dt^2}, \frac{d^2\gamma}{dt^2}\right\rangle = \left(\frac{du}{dt}\right)^4 \left\langle \frac{d^2\gamma}{du^2}, \frac{d^2\gamma}{du^2}\right\rangle.$$

In particular, when $\langle \frac{d^2\gamma}{du^2}, \frac{d^2\gamma}{du^2} \rangle \geq 0$ *we have*

$$\sqrt[4]{\left\langle \frac{d^2\gamma}{du^2}, \frac{d^2\gamma}{du^2} \right\rangle} du = \sqrt[4]{\left\langle \frac{d^2\gamma}{dt^2}, \frac{d^2\gamma}{dt^2} \right\rangle} dt.$$

Proof. Let us denote $\frac{d}{du}$ by putting $'$ and $\frac{d}{dt}$ by $\cdot$. Then we have

$$\begin{aligned} \dot{\gamma} &= \frac{du}{dt}\gamma' \\ \ddot{\gamma} &= \frac{d}{dt}\dot{\gamma} = \frac{du}{dt} \cdot \frac{d}{du}\left(\frac{du}{dt}\gamma'\right) = \frac{du}{dt}\left\{\left(\frac{du}{dt}\right)'\gamma' + \frac{du}{dt}\gamma''\right\}. \end{aligned}$$

Since $\langle \gamma', \gamma' \rangle = 0$ and hence $\langle \gamma', \gamma'' \rangle = 0$ we have $\langle \ddot{\gamma}, \ddot{\gamma} \rangle = \left(\frac{du}{dt}\right)^4 \langle \gamma'', \gamma'' \rangle$, which completes the proof. □

6.3 Characterization of a curve of the osculating circles

We identify the set $\mathcal{S}(1,3)$ of the oriented circles in $\boldsymbol{S}^3$ with the submanifold $\Theta(1,3)$ of $\mathbb{R}^{10}_4$ as before.

Theorem 6.7 *A curve* γ *in* $\mathcal{S}(1,3)$ *is a set of the osculating circles to a curve in* $\boldsymbol{S}^3$ *if and only if* γ *satisfies the following two conditins:*

(i) γ *is a null curve,*

(ii) $\gamma'(s)$ *satisfies the Plücker relations for all* s.

Proof. (1) "Only if" part. The condition (i) follows from Theorem 6.1, and the condition (ii) from the formula (15) as it means that γ' is a pure 3-vector and therefore satisfies the Plücker relations.

(2) "If" part. Lemma 5.1 implies that γ' corresponds to an isotropic 3-space tangent to the light cone. Therefore it defines a point in $\mathbb{E}^3_0$. Let us denote it by

$$\bar{m}(s) = \left(1 + \frac{m(s)\cdot m(s)}{4}, -1 + \frac{m(s)\cdot m(s)}{4}, m(s)\right),$$

and put $C = \{m(s)\}$. We may assume without loss of generality that s is the arc-length of C.

Lemma 6.8 *The point* $m(s)$ *belongs to the circle* $\Gamma(s)$ *which corresponds to* $\gamma(s)$.

Proof. We may assume, without loss of generality, that $\Gamma(s)$ can be obtained as the intersection of our model of $\mathbf{S}^3$ or $\mathbb{E}^3$ in the light cone and span$\langle \boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2 \rangle$. Then, a computation shows that if γ' satisfies the two conditions (i) and (ii) then there is a *null pencil* $\mathcal{P}$ through $\gamma(s)$ (i.e. a 1-parameter family of the oriented circles that are contained in a 2-sphere which contains $\Gamma(s)$ and that are all tangent to $\Gamma(s)$ at a point) such that $\gamma'(s)$ is equal to a tangent vector to $\mathcal{P}$ at $\gamma(s)$ (see [La-OH2] for pencils). $\square$

We have two cases.

(2-a) Suppose $m''(s) \neq \mathbf{0}$. Put

$$\bar{n} = \Big(\frac{m \cdot n}{2}, \frac{m \cdot n}{2}, n\Big) \in \mathbb{R}^5_1 \quad \text{where} \quad n = \frac{m' \times m''}{\|m' \times m''\|} \in \mathbb{R}^3.$$

Then $\bar{m}, \bar{m}', \bar{m}'', \bar{n}$, and $n_1 = (1,1,0,0,0)$ are linearly independent in $\mathbb{R}^5_1$. Therefore, 10 pure 3-vectors obtained as exterior products of three of $\bar{m}, \bar{m}', \bar{m}'', \bar{n}$, and n_1 are linearly independent in $\mathbb{R}^{10}_4$. Express $\gamma(s)$ and $\gamma'(s)$ as the linear combination of them. By the above Lemma, $\gamma(s)$ can be expressed as the linear combination of $\bar{m}\wedge\bar{m}'\wedge\bar{m}''$, $\bar{m}\wedge\bar{m}'\wedge\bar{n}$, $\bar{m}\wedge\bar{m}'\wedge n_1$, $\bar{m}\wedge\bar{m}''\wedge\bar{n}$, $\bar{m}\wedge\bar{m}''\wedge n_1$, and $\bar{m}\wedge\bar{n}\wedge n_1$.

If one of the three coefficients of the latter three does not vanish, there is a non-zero coefficient of $\bar{m}'\wedge u_1\wedge v_1$ ($\{u_1, v_1\} \subset \{\bar{m}'', \bar{n}, n_1\}$) of $\gamma'(s)$, which contradicts to the fact that $\gamma'(s)$ is a pure 3-vector of the form $\bar{m}\wedge u_2\wedge v_2$ ($u_2, v_2 \in \mathbb{R}^5_1$).

Therefore, $\Gamma(s)$ is tangent to C at $m(s)$, i.e. $\gamma(s)$ is of the form

$$\gamma(s) = \xi(s)\,\bar{m}\wedge\bar{m}'\wedge\bar{m}'' + \eta(s)\,\bar{m}\wedge\bar{m}'\wedge\bar{n} + \zeta(s)\,\bar{m}\wedge\ 1zccbarm'\wedge n_1.$$

Since $\langle\gamma'(s), \gamma'(s)\rangle = 0$ we have $\eta(s) = \zeta(s) = 0$. Then, $\langle\gamma(s), \gamma(s)\rangle = 1$ implies $\xi(s) = 1$, which completes the proof.

(2-b) Suppose $m''(s) = \mathbf{0}$. Let u_1 and u_2 be two vectors in $\mathbb{R}^3$ such that $m'(s), u_1$ and u_2 form an orthonormal basis of $\mathbb{R}^3$. Put $\bar{u}_1 = \Big(\frac{m \cdot u_1}{2}, \frac{m \cdot u_1}{2}, u_1\Big)$ and $\bar{u}_2 = \Big(\frac{m \cdot u_2}{2}, \frac{m \cdot u_2}{2}, u_2\Big)$. Then $\bar{m}, \bar{m}', \bar{u}_1, \bar{u}_2$, and $n_1 = (1,1,0,0,0)$ are linearly independent in $\mathbb{R}^5_1$. The rest of the proof goes parallel to the previous case (2-a). Since $\langle\gamma(s), \gamma(s)\rangle = 1$ and $\langle\gamma'(s), \gamma'(s)\rangle = 0$ we have $\gamma = \frac{1}{2}\,\bar{m}\wedge\bar{m}'\wedge n_1$, which means $\gamma = \bar{m}\wedge\bar{m}'\wedge\bar{m}''$, which completes the proof. $\square$

The authors thank Fran Burstall for verifying the theorem by giving an equivalent condtion stated below:

Lemma 6.9 (Burstall; personal communication via Martin Guest) *Suppose* γ *corresponds to a timelike* 3*-space* Π *of* $\mathbb{R}^5_1$. *Recall that* $T_\gamma\mathcal{S}(1,3)$ *can be identified with* $Hom(\Pi, \Pi^\perp)$. *Suppose* γ' *corresponds to an element* A *of* $Hom(\Pi, \Pi^\perp)$. *Then* γ *is a curve of the osculating circles to a curve in* $\boldsymbol{S}^3$ *if and only if* A^tA *vanishes.*

6.4 Conformal arc-length and conformal angles

The conformal arc-length can be interpreted in terms of the conformal angle as follows.

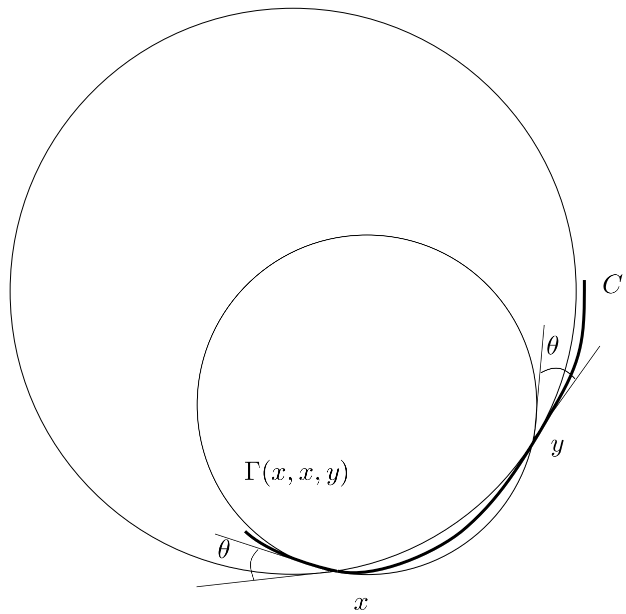

Figure 6: The conformal angle $\theta_C(x,y)$.

Definition 6.10 (Doyle and Schramm) Let x, and y be a pair of distinct points on a curve C. Let $\theta_C(x,y)$ $(0 \le \theta_C(x,y) \le \pi)$ be the angle between $\Gamma(x,x,y)$ and $\Gamma(x,y,y)$. We call it the *conformal angle* between x and y.

We note that the conformal angle is conformally invariant because it can be defined by angles, circles, and tangency, which are preserved by Möbius transformations. Applying Bouquet's formula to $\sin\theta_C$ we have

Lemma 6.11 ([La-OH]) *Let s, κ, τ be the arc-length, curvature, and torsion of C respectively. Then the conformal angle satisfies*

$$\theta_C(x,y) = \frac{\sqrt{{\kappa'}^2 + \kappa^2\tau^2}}{6}\,|x-y|^2 + O(|x-y|^3). \tag{20}$$

The formula (1) of the conformal arc-length implies:

Proposition 6.12 *The conformal arc-length ρ satisfies*

$$\frac{d\rho}{ds}(s) = \lim_{\Delta s\to 0}\frac{\sqrt{6\,\theta_C\big(m(s), m(s+\Delta s)\big)}}{\Delta s}.$$

Remark: Put

$$\Pi = Span\langle \gamma(s,s,s+\Delta s), \gamma(s,s+\Delta s,s+\Delta s)\rangle.$$

It is a spacelike 2-plane of $\mathbb{R}^{10}_4$. The intersection of Π and $\Theta(1,3)$ is a circle which becomes a geodesic of $\Theta(1,3)$. It consists of the oriented circles that pass through both $m(s)$ and $m(s+\Delta s)$ and that are contained in the sphere which contains the two tangent circles $\Gamma(m(s),m(s),m(s+\Delta s))$ and $\Gamma(m(s),m(s+\Delta s),m(s+\Delta s))$. Then the conformal angle is equal to the distance in $\mathcal{S}(1,3)$ between $\gamma(s,s,s+\Delta s)$ and $\gamma(s,s+\Delta s,s+\Delta s)$.

Remark: We have two kinds of infinitesimal interpretation of the conformal arc-length as the distance between a pair of nearby circles: one by osculating circles $\Gamma(x,x,x)$ and $\Gamma(y,y,y)$ (Theorem 6.3) and the other by tangent circles $\Gamma(x,x,y)$ and $\Gamma(x,y,y)$ (Proposition 6.12). We remark that the former is a "timelike" pair and the latter a "spacelike" pair.

6.5 Conformal arc-length as unitary curvature parameter

Let us introduce another characterization of the conformal arc-length. Since the curve γ consisting of the osculating circles is a null curve in $\Theta(1,3)$, we have $\langle \gamma,\gamma\rangle \equiv 1$ and $\langle \dot{\gamma},\dot{\gamma}\rangle \equiv 0$ for any parameter t of the original curve C. Corollary 6.4 and Lemma 6.6 imply

Proposition 6.13 *The conformal arc-length ρ can be characterized as the parameter that satisfies $\langle \ddot{\gamma}, \ddot{\gamma} \rangle \equiv 1$.*

The reader is referred to [Ak-Go], [Fi], [HJ], [MRS], [Ro-Sa], and [Su1, Su2] for further details in conformal differential geometry. A conformally invariant moving frame is given in [Yu].

7 Other viewpoints

7.1 Information from two nearby osculating circles

Definition 7.1 ([La-OH]) Let P_1,P_2,P_3, and P_4 be points on an oriented sphere Σ. They can be considered as complex numbers through an orientation preserving stereographic projection from Σ to $\mathbb{C}\cup\{\infty\}$. The cross ratio of P_1,P_2,P_3, and P_4 can be defined by that of the corresponding four complex numbers. We remark that it does not depend on a stereographic projection. (The reader is referred to [O'H1] for the details.)

Let C be a curve in $\mathbf{S}^3$ or $\mathbb{R}^3$, and $\mathcal{S}$ be a sphere which intersects C orthogonally at a point $m(t)$ that passes through a nearby point $m(t+h)$. The osculating circles $\mathcal{O}_{m(t)}$ and $\mathcal{O}_{m(t+h)}$ to C intersect $\mathcal{S}$ at four points $m(t), y_1 \in \mathcal{O}_{m(t)}$ and $m(t+h), y_2 \in \mathcal{O}_{m(t+h)}$.

Proposition 7.2 *The infinitesimal cross ratio $(m(t),y_2;m(t+dt),y_1)$ is real. The fourth root of its absolute value is equal to the conformal arc-length of the curve C.*

Proof. Let us consider the osculating spheres $\Sigma_{m(t)}$ and $\Sigma_{m(t+h)}$ to the curve C at $m(t)$ and $m(t+h)$. The angle between them is of the order of h. The distance between the point $m(t+h)$ and $\Sigma_{m(t)}$ and the distance between the point $m(t)$ and $\Sigma_{m(t+h)}$ are of the order $|h|^4$. The two osculating spheres $\Sigma_{m(t)}$ and $\Sigma_{m(t+h)}$ meet the small sphere $\mathcal{S}$ in two almost totally geodesic circles Γ_1 and Γ_2. The two circles meet at two points A and B which are almost antipodal on $\mathcal{S}$. Suppose A is close to $m(t)$ and y_2, and B to $m(t+h)$ and y_1 as in Figure

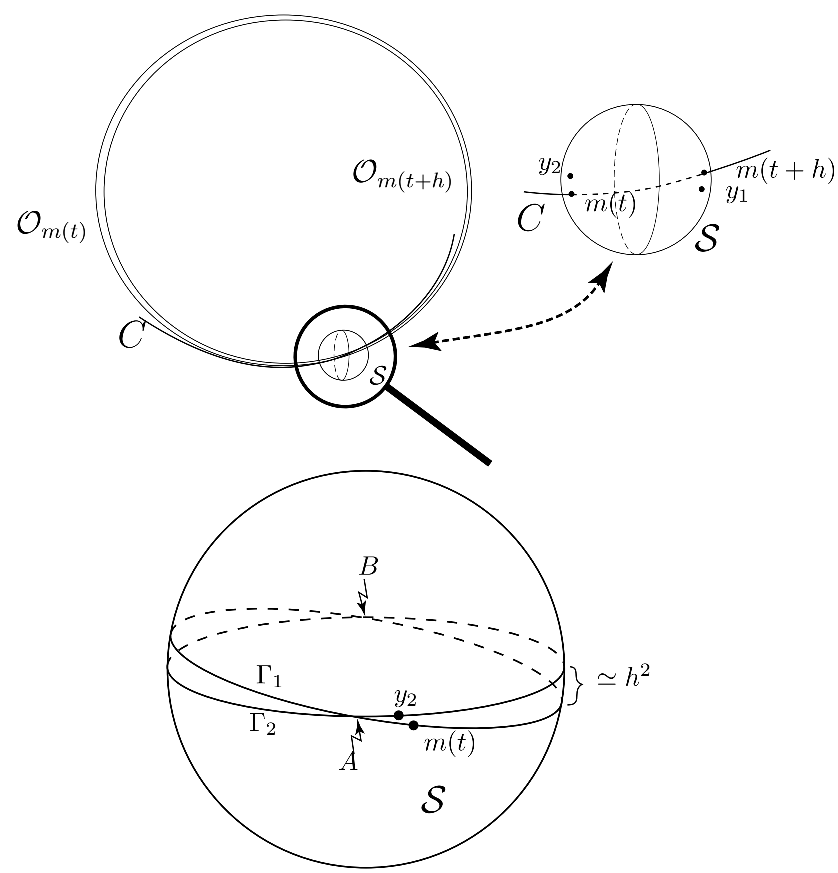

Figure 7: Trace of the osculting circles and of the two osculating spheres at two nearby points of C on an almost orthogonal sphere $\mathcal{S}$

7. The distance between $m(t)$ and y_2 and the distance between $m(t+h)$ and y_1 are of order at most $|h|^3$ as they are equivalent to distances between $m(t)$ and $\mathcal{O}_{m(t+h)}$ and $m(t+h)$ and $\mathcal{O}_{m(t)}$. As the angle between the osculating spheres $\mathcal{O}_{m(t)}$ and $\mathcal{O}_{m(t+h)}$ is of the order h, so is the angle between the circles Γ_1 and Γ_2. Therefore, the distances between A and $m(t)$, A and y_2, B and $m(t+h)$, and B and y_1 are of the order smaller than the diameter of the sphere $\mathcal{S}$ multiplied

by h, which means that they are negligeable compared with h^2 (see Figure 7).

It implies that the distance between $m(t)$ or y_1 and Γ_2, and that between $m(t+h)$ or y_2 and Γ_1 is negligible compared with $|h|^2$ (see Figure 7).

After renormalizing the sphere $\mathcal{S}$ by a suitable homothety to get a sphere $\tilde{\mathcal{S}}$ of radius 1, the four points $\tilde{m}(t), \tilde{y}_2, \tilde{m}(t+h), \tilde{y}_1$ are almost on a circle, therefore have a cross-ratio which is almost real. A computation shows that this cross ratio is of order greater or equal to the order of $|h|^4$. We know that the conformal arc-lenth of a curve $C \subset \mathbb{R}^3$ parametrized by the arc-length is $\omega = \sqrt{\psi} ds$ where $\psi = \sqrt{(k')^2 + k^2\tau^2}$. Comparing with the normal form (see [CSW])

$$\begin{array}{l} y = \frac{x^3}{3!} + O(x^5), \\ z = 0 + O(x^4), \end{array} \tag{21}$$

where at $x = 0$ the derivative of the curvature is 1, and as the absolute value of our cross-ratio is equivalent to the square of the distance of the point $m(t+h)$ to the osculating circle to C at $m(t)$, divided by $|h|^2$, the proposition is proven. $\square$

The construction of the normal form in [CSW] is analytical. Nevertheless we can see from the form of the equations (21) that:

- the osculating circle at the origin in the model is a line
- the osculating sphere at the origin in the model is the (x, y)-plane
- the derivative of the curvature of the curve at the origin is 1

We have not found the geometric interpretation of the absence of the x^4 term is the expression of y.

7.2 Conformal arc-length starting from the lightlike curves in the set of spheres associated to a given curve in $\mathbf{S}^3$ or $\mathbb{R}^3$

Let C be a vertex-free curve in $\mathbf{S}^3$. Let Γ be a curve in Λ^4 which is the set of osculating spheres $\sigma(t)$ of C. Let $\tilde{s}$ be the arc-length of Γ, and $\bullet$ denote $\frac{d}{d\tilde{s}}$. Then Γ is a *drill* (see [La-So]), that is, its geodesic curvature vector $\boldsymbol{k_g} = \ddot{\sigma} + \sigma$ is lightlike at each point.

From it we can construct a surface $V(C)$ in Λ^4 as the union of the geodesic circles tangent to Γ. The surface $V(C)$ can also be seen as the set of spheres which contain an osculating circle to C. The curves orthogonal to the "foliation" of $V(C)$ by these geodesic circles are lightlike and generically have cuspidal edges at points of Γ. To see that, consider two nearby geodesic circles tangent to Γ: $\Lambda^4 \cap \operatorname{span}\langle \sigma(s), \dot{\sigma}(s)\rangle$ and $\Lambda^4 \cap \operatorname{span}\langle \sigma(s+ds), \dot{\sigma}(s+ds)\rangle$. It follows that the tangent space to $V(C)$ at a point $\nu(s,\theta) = \cos\theta\sigma(s) + \sin\theta\dot{\sigma}(s)$ is given by

$$T_\nu\Lambda^4 \cap \operatorname{span}\langle \sigma(s), \dot{\sigma}(s), \ddot{\sigma}(s)\rangle = T_\nu\Lambda^4 \cap \operatorname{span}\langle \sigma(s), \dot{\sigma}(s), k_g(\sigma(s))\rangle.$$

Notice that the angle between two spheres $\nu(s,\theta)$ and $\nu(s,\theta')$ is independant of the value of s.

The normal space in $T_{\nu(s,\theta)}V(C)$ to the geodesic circle $\{\cos\theta\,\sigma(s)+\sin\theta\,\overset{\bullet}{\sigma}(s)\}$ through $\nu(s,\theta)$ is therefore generated by $\boldsymbol{k_g}(\sigma(s))$. For each fixed θ_0, the curve $\nu(s,\theta_0)$ is a lightlike curve.

Theorem 7.3 *Let C be a vertex-free curve. If we take the average of the $\frac{1}{2}$-dimensional measures of the lightlike curves through $\cos\theta\,\sigma(t)+\sin\theta\,\overset{\bullet}{\sigma}(t)$ by moving θ, we get, up to the multiplication by a universal constant, the conformal arc-length of the curve C.*

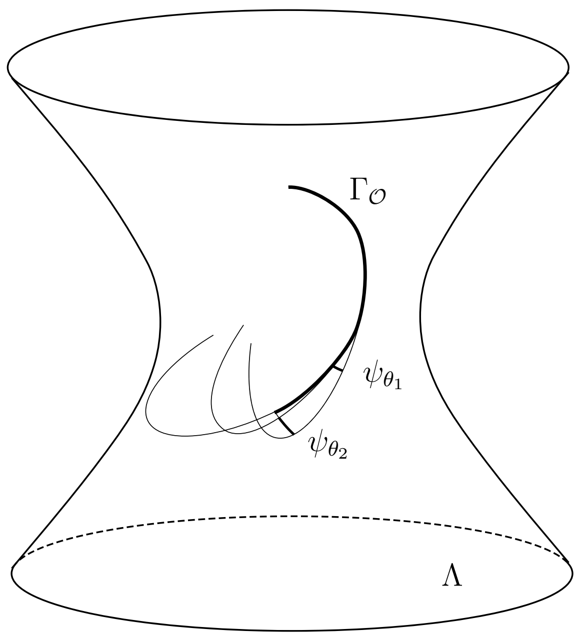

Figure 8: Lightlike curves made of spheres containing the osculating circles

The statement is analogous to the following metric statement:

Proposition 7.4 *The curvature of a curve $C \subset \mathbb{R}^3$ at a point m is proportional to the average of the curvatures at m of the planes curves obtained as the orthogonal projections of C on the planes containing the tangent line T_mC to C at m (the proportionality coefficient is π).*

In order to prove the proposition 7.4 we only need to project an osculating circle $\mathcal{O}_m$ to C at m on the planes containing T_mC and observe the curvatures of these projections at the point m.

Definition 7.5 By *Lorentz distance* between a pair of spheres Σ_1 and Σ_2 we mean the length of the arc of the geodesic joining the two correspoinding points

σ_1 and σ_2 in Λ: $d_{\mathcal{L}}(\sigma_1, \sigma_2) = \int \|\dot{\gamma}\| dt$, where the geodesics of Λ are the intersection of Λ with vectorial 2-planes.

Before proving Theorem 7.3 we need two lemmas to compare the Lorentz distance between a pair of spheres to the Lorentz distance between their intersections with an orthogonal sphere or an orthogonal circle, or with a sphere or a circle almost orthogonal to them.

Lemma 7.6 (1) *Let S be a sphere orthogonal to two spheres Σ_1 and Σ_2. Then the Lorentz distance $d_{\mathcal{L}}(\sigma_1, \sigma_2)$ between Σ_1 and Σ_2 is equal to the Lorentz distance $d_{\mathcal{L}}(\gamma_1, \gamma_2)$ between the two circles $\Gamma_1 = \Sigma_1 \cap S$ and $\Gamma_2 = \Sigma_2 \cap S$.*

(2) *Let $\mathcal{G}$ be a circle orthogonal to two spheres Σ_1 and Σ_2. Then the Lorentz distance $d_{\mathcal{L}}(\sigma_1, \sigma_2)$ is equal to the Lorentz distance $d_{\mathcal{L}}(\mathcal{P}_1, \mathcal{P}_2)$ between the two 0-spheres $\mathcal{P}_1 = \Sigma_1 \cap \mathcal{G}$ and $\mathcal{P}_2 = \Sigma_2 \cap \mathcal{G}$.*

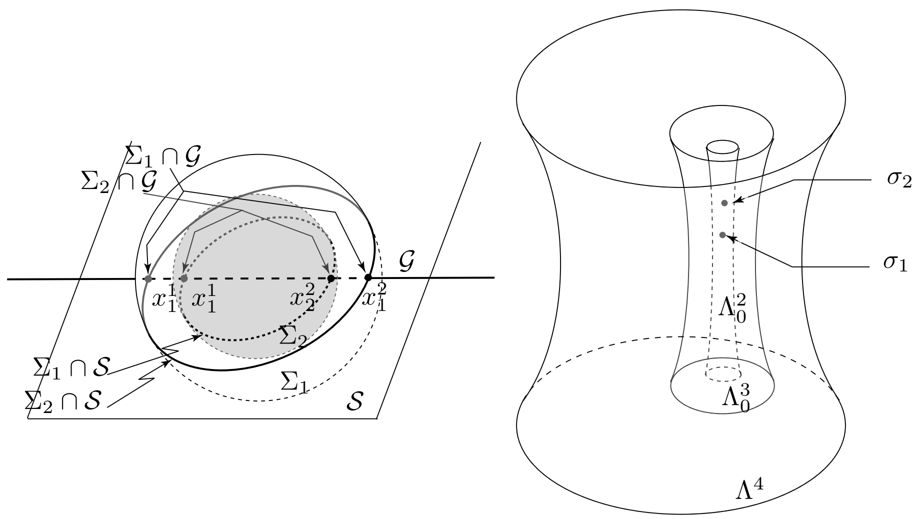

Figure 9: Intersection and Lorentz distance

Proof. The two items can be proven by considering the two points σ_1 and σ_2 corresponding to the two spheres Σ_1 and Σ_2 in de Sitter spaces $\Lambda_0^2 \subset \Lambda^4$ or $\Lambda_0^3 \subset \Lambda^4$, where Λ_0^2 is the set of the oriented spheres orthogonal to a circle $\mathcal{G}$ orthogonal two both spheres Σ_1 and Σ_2, and Λ_0^3 is the set of the spheres orthogonal to the sphere S. □

Remark: The Lorentz distance between Σ_1 and Σ_2 and the cross-ratio of the four intersection points of $\mathcal{G}$ and $\Sigma_1 \cup \Sigma_2$ are related by a diffeomorphism. When the circle $\mathcal{G}$ is a line the cross-ration is given by the formula:

$$cross(x_1^1, x_2^1; x_1^2, x_2^2) = \frac{x_1^1 - x_2^1}{x_1^1 - x_2^2} : \frac{x_1^2 - x_2^1}{x_1^2 - x_2^2}.$$

When the four points are on a circle in a complex plane, the four points x_i^j should be considered as complex numbers; the result is real as the points are on a circle.

For example if we name the point as in Figure (9), we have:

$$|cross(x_1^1, x_2^1; x_1^2, x_2^2)| = \left(\frac{e^\ell - 1}{e^\ell + 1}\right)^2,$$

where ℓ is the Lorentz distance $d_{\mathcal{L}}(\sigma_1, \sigma_2)$.

In particular, when the Lorentz distance between Σ_1 and Σ_2 is small, we have:

$$|cross(x_1^1, x_2^1; x_1^2, x_2^2)| \simeq \left(\frac{\ell}{2}\right)^2.$$

Lemma 7.7 *Suppose that a circle $\bar{\mathcal{G}}$ makes angles θ_1 with Σ_1 and θ_2 with Σ_2, then the quotient of the cross ratios $cross(x_1^1, x_2^1; x_1^2, x_2^2)$ of the intersection of Σ_1 and Σ_2 with a circle $\mathcal{G}$ orthogonal to both spheres and the cross ratio $cross(y_1^1, y_2^1; y_1^2, y_2^2)$ of the intersection with $\bar{\mathcal{G}}$ is in an interval of the form $[1-\delta_1, 1+\delta_1]$, where δ_1 is a function of θ_1 and θ_2 which goes to 0 when θ_1 and θ_2 go to $\pm\pi/2$.*

Suppose that the sphere S makes angles θ_1 with Σ_1 and θ_2 with Σ_2, then the quotient of the cross ratios of the intersection $\Sigma_1 \cap S$ and $\Sigma_2 \cap S$ with a common orthogonal circle $\gamma \subset S$ and the cross ratio $cross(x_1^1, x_2^1; x_2^1, x_2^2)$ of the intersection of Σ_1 and Σ_2 with a circle $\mathcal{G}$ orthogonal to both spheres is in an interval of the form $[1-\delta_2, 1+\delta_2]$, where δ_2 is a function of θ_1 and θ_2 which goes to 0 when θ_1 and θ_2 go to $\pm\pi/2$.

Proof. Let us fix the two spheres Σ_1 and Σ_2.

Let us consider the circle $\tilde{\mathcal{G}}$ orthogonal to Σ_1 at y_1^1 and y_2^1. It intersects $\mathcal{G}$ at y_1^1 and y_2^1 making a small angle $\pi/2 - \theta_1$.

It also intersects Σ_2 at points z_1 and z_2 close to y_1^2 and y_2^2 and with an angle close to $\pi/2$. All the corresponding arcs a_i, b_i on both circles have a ratio satisfying $1-\delta < a_i/b_i < 1+\delta$, which implies the first statement of the lemma.

The second statement of the lemma comes from the fact that the circle γ is almost orthogonal to the spheres Σ_2 and Σ_2. □

Proof of Theorem7.3: Let $\Gamma_{\mathcal{O}} \subset \Lambda^4$ be the curve of osculating spheres to the curve C, and $\sigma(\tilde{s})$ the osculating sphere at the point $m(\tilde{s})$ to the curve $C = \{c(\tilde{s})\}$, where $\tilde{s}$ is an arc-lenth parameter on the curve $\Gamma_{\mathcal{O}} \subset \Lambda^4$.

We remark that $\Gamma_{\mathcal{O}}$ is spacelike if C is vertex-free.

Proposition 7.8 *Let C be a vertex-free curve, $\overline{\Sigma}_0$ a sphere which has second order contact with C, and $\bar{\sigma}_0$ a point in Λ corresponding to $\overline{\Sigma}_0$. Then there is a unique lightlike curve $\bar{\sigma}$ through $\bar{\sigma}_0$ consisting of the spheres $\overline{\Sigma}$ with second order contact with C.*

The statement without "*uniqueness*" was proved in Corollary 10 of [La-So].

Proof. Every sphere having second order contact at a point $m(\tilde{s})$ in C can be written by

$$\cos\theta\,\sigma(\tilde{s}) - \sin\theta\,\overset{\bullet}{\sigma}(\tilde{s})$$

for some $\theta \in [0, 2\pi)$, where $\bullet$ denotes $d/d\tilde{s}$. Let $\eta(\tilde{s})$ be a curve in Λ given by

$$\eta(\tilde{s}) = \cos u(\tilde{s})\,\sigma(\tilde{s}) - \sin u(\tilde{s})\,\overset{\bullet}{\sigma}(\tilde{s}),$$

where $u(\tilde{s})$ is a function. Then we have

$$\overset{\bullet}{\eta} = \cos u\,\big(1 - \overset{\bullet}{u}\big)\,\overset{\bullet}{\sigma} - \sin u\,\big(\overset{\bullet}{u}\sigma + \overset{\bullet\bullet}{\sigma}\big).$$

As $\langle\sigma,\sigma\rangle = \langle\overset{\bullet}{\sigma},\overset{\bullet}{\sigma}\rangle = 1$ we have $\langle\overset{\bullet}{\sigma},\sigma\rangle = \langle\overset{\bullet}{\sigma},\overset{\bullet\bullet}{\sigma}\rangle = 0$. Furthermore we have $\langle\overset{\bullet\bullet}{\sigma},\overset{\bullet\bullet}{\sigma}\rangle = 1$ ([La-So],[Yu]). Therefore we have

$$\langle\overset{\bullet}{\eta},\overset{\bullet}{\eta}\rangle = \big(1 - \overset{\bullet}{u}\big)^2,$$

which implies that $\eta(\tilde{s})$ is lightlike if and only if $u(\tilde{s}) = \tilde{s} + \theta$ for some constant θ.

If $\bar{\sigma}_0$ can be expressed as

$$\bar{\sigma}_0 = \cos u_0\,\sigma(\tilde{s}_0) - \sin u_0\,\overset{\bullet}{\sigma}(\tilde{s}_0)$$

then $\bar{\sigma}$ is uniquely determined by

$$\bar{\sigma}(\tilde{s}) = \cos\left(\tilde{s} + u_0 - \tilde{s}_0\right)\sigma(\tilde{s}) - \sin\left(\tilde{s} + u_0 - \tilde{s}_0\right)\overset{\bullet}{\sigma}(\tilde{s}),$$

which has the following lightlike tangent vector

$$\overset{\bullet}{\bar{\sigma}}(\tilde{s}) = -\sin(\tilde{s} + u_0 - \tilde{s}_0)\big(\sigma(\tilde{s}) + \overset{\bullet\bullet}{\sigma}(\tilde{s})\big). \tag{22}$$

□

Remark: Let

$$\psi(\theta,\tilde{s}) = \cos(\tilde{s}+\theta)\sigma(\tilde{s}) - \sin(\tilde{s}+\theta)\overset{\bullet}{\sigma}(\tilde{s}),$$

and ψ_θ be a lightlike curve $\{\psi(\theta,\ \bullet\ )\}$ in Λ^4. Using one of the curves ψ_θ we can construct a surface M_θ containing C such that C is a line of principal curvature of M_θ. Then the spheres of ψ_θ are the osculating spheres to M_θ at points of C. Any surface tangent to M_θ along C has the same property.

When C is contained in $\mathbb{R}^3$ a developable M_θ is obtained as the envelopes of planes $P_\theta(\tilde{s}+h)$ satisfying:

- the angle between $P_\theta(\tilde{s})$ and the osculating plane to C at $c(\tilde{s})$ is θ,
- the derivative, with respect to an arc-lenth parameter s on the curve C, of the angle between $P_\theta(\tilde{s}+h)$ and the osculating plane to C at $c(\tilde{s}+h)$ is $-\tau(\tilde{s}+h)$, the opposite of the torsion of the curve C at $c(\tilde{s}+h)$.

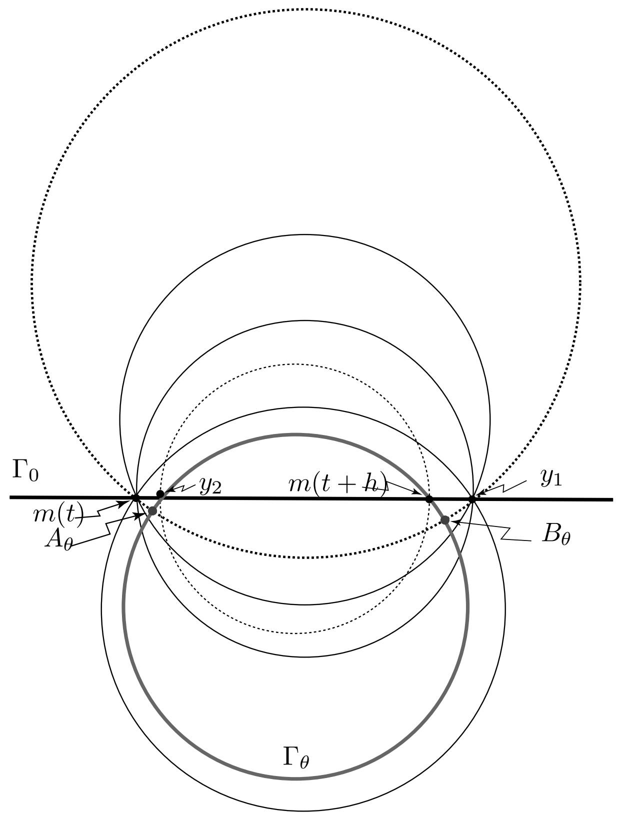

Figure 10: Trace of the spheres $\psi_\theta(s)$ and $\psi_\theta(s+h)$ on an almost orthogonal sphere S (with a point of S at infinity)

Corollary 7.9 *The integral of the $L^{1/2}$-length element of the lightlike curves ψ_θ is* $(\pi/12)|\mathcal{L}(\dot{\sigma} + \dddot{\sigma})|$.

Proof. Computing the second derivative of ψ_θ, we get:

$$\ddot{\psi}_\theta = -\cos(\tilde{s}+\theta)\left(\sigma(\tilde{s}) + \ddot{\sigma}(\tilde{s})\right) - \sin(\tilde{s}+\theta)\left(\dot{\sigma}(\tilde{s}) + \dddot{\sigma}(\tilde{s})\right).$$

As $\sigma(\tilde{s}) + \ddot{\sigma}(\tilde{s})$ is lightlike and orthogonal to its derivative $\dot{\sigma}(\tilde{s}) + \dddot{\sigma}(\tilde{s})$, we have

$$\frac{1}{12}\int_{\mathbf{S}^1} \left|\mathcal{L}(\ddot{\psi}_\theta)\right| d\theta = \frac{1}{12}\int_{\mathbf{S}^1} (\sin(\tilde{s}+\theta))^2 |\mathcal{L}(\dot{\sigma}(\tilde{s}) + \dddot{\sigma}(\tilde{s}))|\, d\theta = \frac{\pi}{12}|\mathcal{L}(\dot{\sigma} + \dddot{\sigma})|.$$

□

(Proof of Theorem 7.3 continued)

Let S be the sphere orthogonal to C at $m(s)$ which also contains the point $m(s+h)$. We will now follow the intersection with S of the spheres of the different lightlike curves ψ_θ, where θ is the angle of the initial sphere of the family $\psi_\theta(t_0)$ with a chosen sphere of contact of second order with C at the point $m_0 = c(s_0)$.

Consider two points $m(s)$ and $m(s+h)$ on C (here we use the arc-length parameter on the curve $C \subset \mathbf{S}^3$). The set of the spheres $\{\psi(\theta, s) \,|\, \theta \in \mathbf{S}^1\}$ and $\{\psi(\theta, s+h) \,|\, \theta \in \mathbf{S}^1\}$ are two pencils consisting of the spheres which contain $\mathcal{O}_{m(s)}$ and $\mathcal{O}_{m(s+h)}$ respectively. They intersect the sphere S in two pencils of circles with base points $\{m(s), y_1\}$ and $\{y_2, m(s+h)\}$ (see Figure 7).

As the sphere S is orthogonal to the osculating circle to the curve C at $m(s)$, which is the base circle of the pencil $\{\psi(\theta, s) \,|\, \theta \in \mathbf{S}^1\}$, the angle θ is also the angle parameter of the pencil of circles $\{\psi(\theta, s) \cap S \,|\, \theta \in \mathbf{S}^1\}$. The angle of the circles $S \cap \psi(\theta, s+h)$ and $S \cap \psi(\theta', s+h)$ is only close to $\theta - \theta'$ as we know that the angle of S and $\mathcal{O}_{m(s+h)}$ is close to $\pi/2$.

Let Γ_0 denote the circle of S containing the three points $m(s), y_1$ an $(s+h)$. As the cross-ratio $cross(m(s), y_2, m(s+h), y_1)$ is almost real, the fourth point y_2 is almost on Γ_0 (that is, still would be so after performing homothety which makes the radius of S being equal to 1). Therefore the cross ratio $cross(m(s), y_2, m(s+h), y_1)$ is equivalent to the modulus of the two circles $S \cap \psi(\theta_1, s)$ and $S \cap \psi(\theta_1, s+h)$, where the value θ_1 is such that $S \cap \psi_{\theta_1}(s)$ is orthogonal to Γ_0. Let us now consider the intersection of the two circles $S \cap \psi(\theta, s)$ and $S \cap \psi(\theta, s+h)$ with the circle Γ_θ which is orthogonal to $S \cap \psi(\theta, s)$ (see Figure 7). The modulus of these two circles is equivalent to the cross-ratio

$$cross(m(s), A_\theta; B_\theta, y_1) \simeq \cos^2(\theta - \theta_1)\, cross(m(s), y_2, m(s+h), y_1).$$

Using Lemma 7.7, we see that the modulus of the spheres $\psi_\theta(s)$ and $\psi_\theta(s+h)$ is also equivalent to

$$\cos(\theta - \theta_1)\, cross(m(s), y_2, m(s+h), y_1).$$

The modulus of two close disjoint spheres is equivalent to their Lorentz distance. We can integrate the contribution of the segments $\psi_\theta(t), \psi_\theta(s+h)$ to the $L^{1/2}$-lengthes of the curves $\psi_\theta \subset \Lambda$. It is:

$$\left(\int_{\theta \in \mathbf{S}^1} |\cos(\theta - \theta_1)|^{1/2} d\theta \right) \sqrt{2}\, \sqrt[4]{|cross(m(t), y_2, m(t+h), y_1)|}\,.$$

This provides the constant (not a very enlightening one) in the statement of Theorem 7.3. $\square$

References

[Ak-Go] M. A. Akivis and V. V. Goldberg, *Conformal differential geometry and its generalizations*, Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, xiv+383 pages (1996).

[CSW] G. Cairns, R. W. Sharpe and L. Webb, *Conformal invariants for curves and surfaces in three dimensional space forms*, Rocky Mountain Journal of Math. **24** (1994), 933 – 959.

[Fi] A. Fialkow. The conformal theory of curves. Trans. Amer. Math. Soc. 51 (1942), 435-501.

[HJ] U. Hertrich-Jeromin, *Introduction to Möbius differential geometry*, London Mathematical Society Lecture Note Series, 300. Cambridge University Press, Cambridge, xii+413 pages (2003).

[Ko-Yo] T. Kobayashi and T. Yoshino, *Compact Clifford-Klein forms of symmetric spaces - revisited, Special Issue: In memory of Armand Borel*, Pure Appl. Math. Quart., **1** (2005), 591 – 663.

[La-OH] R. Langevin and J. O'Hara, *Conformally invariant energies of knots,* J. Institut Math. Jussieu 4 (2005), 219-280.

[La-OH2] R. Langevin and J. O'Hara, *Extrinsic Conformal Geometry* $\sim$ *with applications to Curves and Surfaces* $\sim$*,* in preparation.

[La-So] R. Langevin and G. Solanes. *Conformal geometry of curves and the length of canals,* Preprint Dijon (2005).

[Li] H. Liebmann, *Beiträge zur Inversionsgeometrie der Kurven*, Münchener Berichte (Akademie der Wissenschaften, Munich, Sitzungsberichte), (1923) 79 – 94.

[MRS] A. Montesinos Amilibia, M. C. Romero Fuster and E. Sanabria Codesal, *Conformal curvatures of curves in* $\mathbb{R}^{n+1}$, Indag. Mathem. N.S. **12** (3) (2001), 369 – 382.

[O'H1] J. O'Hara, *Energy of knots and conformal geometry,* Series on Knots and Everything Vol. 33, World Scientific, Singapore, xiv + 288 pages.

[O'H2] J. O'Hara, *Energy of knots and the infinitesimal cross ratio,* to appear in Proceedings of the Conference "Groups, Homotopy and Configuration Spaces", Geometry and Topology Monographs.

[O'N] B. O'Neil. *Semi-Riemannian geometry. With applications to relativity*, Pure and Applied Mathematics, 103. Academic Press, Inc., New York, xiii+468 pages (1983).

[Ro-Sa] M. C. Romero-Fuster and E. Sanabria-Codesal, *Generalized evolutes, vertices and conformal invariants of curves in* $\mathbb{R}^{n+1}$, Indag. Mathem. N.S. **10** (2) (1999), 297 – 305.

[Su1] R. Sulanke, *Submanifolds of the Möbius space II, Frenet Formulas and Curves of Constant Curvatures*, Math. Nachr. **100** (1981), 235 – 247.

[Su2] R. Sulanke, *Möbius invariants for pairs* (S_1^m, S_2^l) *of spheres in the Möbius space* S^n, Beiträge zur Algebra und Geometrie.

[Ta] T. Takasu, *Differentialgeometrien in den Kugelräumen, I, Konforme Differentialkugelgeometrie von Liouville und Möbius* , Maruzen, Tokyo (1938).

[Yu] L. Yun. *Invariant of curves under Möbius group in 3-dimensional space* Peking University (2003).

Institut de Mathématiques de Bourgogne, Université de Bourgogne,
CNRS-UMR 5584, U.F.R. Sciences et Techniques, 9, avenue Alain Savary - B.P. 47870, 21078 DIJON Cedex, FRANCE.
E-mail: langevin@u-bourgogne.fr

Department of Mathematics, Tokyo Metropolitan University,
1-1 Minami-Ohsawa, Hachiouji-Shi, Tokyo 192-0397, JAPAN.
E-mail: ohara@tmu.ac.jp