

On the asymptotic behavior of bifurcating autoregressive processes

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Journées Algorithmes Stochastiques

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Outline

- 1 Introduction
 - Bifurcating autoregressive processes
 - The state of the art
- 2 Least squares estimation
 - Martingale assumptions
 - Our least squares estimators
- 3 Main results
 - A keystone lemma
 - Strong laws of large numbers
 - Central limit theorems

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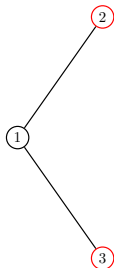
3 Main results

- A keystone lemma
- Strong laws of large numbers
- Central limit theorems

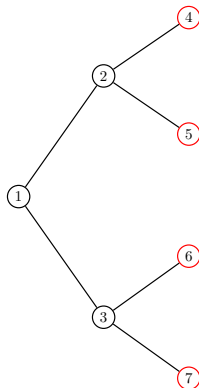
Cell division

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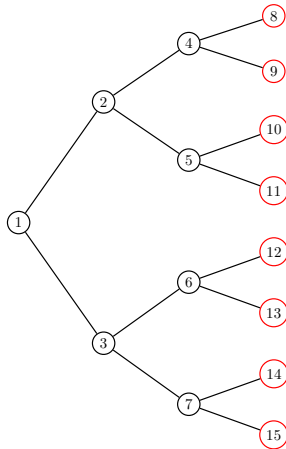
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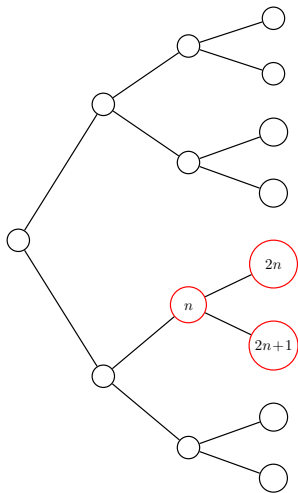
Cell division



Cell division



Ancestors and offsprings



The offsprings of cell n are labelled

- $2n$,
- $2n + 1$.

For each cell n , X_n can be

- the diameter,
- the age,...

Bifurcating AutoRegressive processes

Consider the **BAR**(p) process given, for all $n \geq 2^{p-1}$, by

$$\begin{cases} X_{2n} &= a + \sum_{k=1}^p b_k X_{\lfloor \frac{n}{2^{k-1}} \rfloor} + \varepsilon_{2n}, \\ X_{2n+1} &= c + \sum_{k=1}^p d_k X_{\lfloor \frac{n}{2^{k-1}} \rfloor} + \varepsilon_{2n+1} \end{cases}$$

where $\lfloor x \rfloor$ stands for the integer part of x and $(\varepsilon_{2n}, \varepsilon_{2n+1})$ is the driven noise of the process.

Goal

Statistical inference on the unknown parameters (a, b, c, d) .

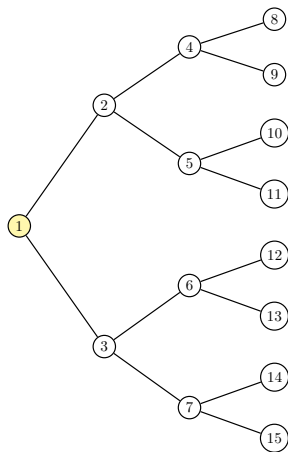
Bifurcating AutoRegressive processes

For simplicity, we focus on the **BAR(1)** process given, for all $n \geq 1$, by

$$\begin{cases} X_{2n} &= a + b X_n + \varepsilon_{2n}, \\ X_{2n+1} &= c + d X_n + \varepsilon_{2n+1} \end{cases}$$

where $0 < \max(|b|, |d|) < 1$ and $|a| + |c| \neq 0$.

Binary tree, generations, subtrees



- Generation 0 is $\mathbb{G}_0 = \{1\}$.
- Generation 1 is $\mathbb{G}_1 = \{2, 3\}$.
- Generation 2 is $\mathbb{G}_2 = \{4, 5, 6, 7\}$.
- Generation n is

$$\mathbb{G}_n = \{2^n, 2^n + 1, \dots, 2^{n+1} - 1\}.$$

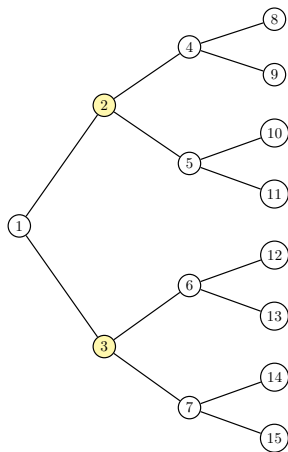
- Subtree up to time n is

$$\mathbb{T}_n = \bigcup_{k=0}^n \mathbb{G}_k.$$

- Cardinalities are $|\mathbb{G}_n| = 2^n$ and

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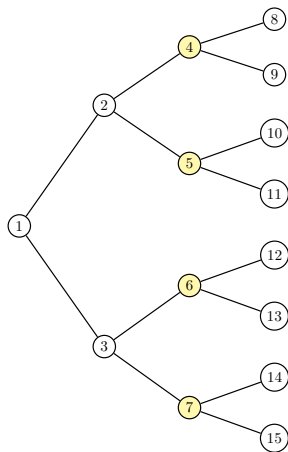
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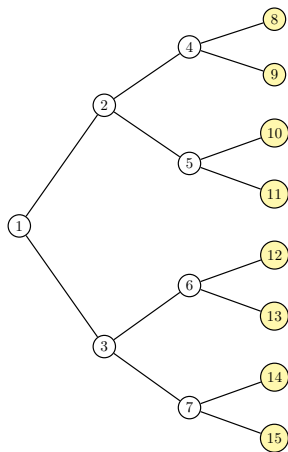
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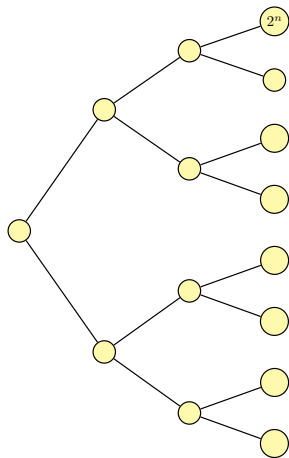
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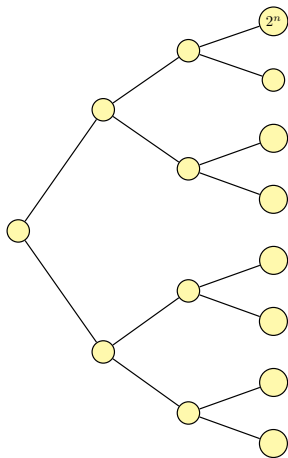
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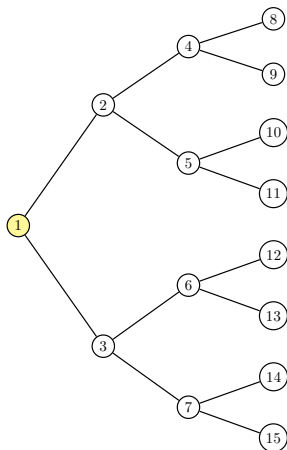
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Filtration



- At time 0, $\mathcal{F}_0 = \sigma\{X_1\}$.

- At time 1, $\mathcal{F}_1 = \sigma\{X_1, X_2, X_3\}$.

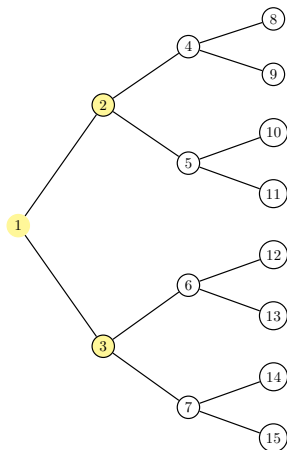
- At time 2,

$$\mathcal{F}_2 = \sigma\{X_1, X_2, X_3, X_4, X_5, X_6, X_7\}.$$

- At time n ,

$$\mathcal{F}_n = \sigma\{X_k \text{ with } k \in \mathbb{T}_n\}.$$

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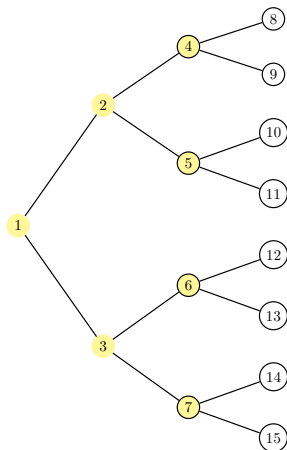
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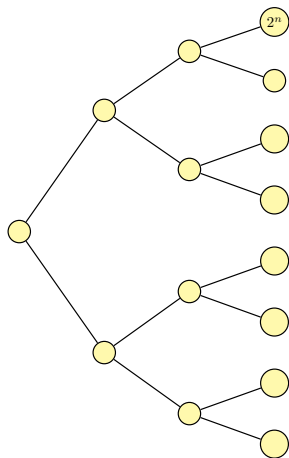
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Previous works I

Stationary processes

- Cowan, Staudte, Biometrics, 1986 : Introduction, motivation in Statistical Biology.
- Huggins, Annals of Statistic, 1996 : Maximum likelihood estimation for large trees.
- Huggins, Basawa, Journal of Applied Probability, 1999 and Australian Journal of Statistics, 2000 : Maximum likelihood estimation for large trees and higher order.
- Basawa, Zhou, Journal of Applied Probability, 2004 : exponential BAR processes, and Journal of Time Series Analysis, 2005 : CLT for BAR processes.

Previous works II

Non stationary processes

- Guyon, Annals of Applied Probability, 2007 : Least square estimation for Gaussian BAR processes via a Markov chain approach.
- Delmas, Marsalle, 2008, 2009 : The same as Guyon and possibility for a cell to die.
- Our work, EJP 2009 : Least square estimation for martingale difference $\text{BAR}(\rho)$ processes by use of a martingale approach.

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Martingale assumptions

(H.1) $\forall n \geq 0$ and $\forall k \in \mathbb{G}_{n+1}$,

$$\mathbb{E}[\varepsilon_k | \mathcal{F}_n] = 0 \quad \text{and} \quad \mathbb{E}[\varepsilon_k^2 | \mathcal{F}_n] = \sigma^2 > 0 \quad \text{a.s.}$$

(H.2) $\forall n \geq 0$ and $\forall k \neq \ell \in \mathbb{G}_{n+1}$,

- if $[k/2] \neq [\ell/2]$, then ε_k and ε_ℓ are independent given \mathcal{F}_n
- if $[k/2] = [\ell/2]$, then for $\rho < \sigma^2$

$$\mathbb{E}[\varepsilon_k \varepsilon_\ell | \mathcal{F}_n] = \rho \quad \text{a.s.}$$

(H.3)

$$\sup_{n \geq 0} \sup_{k \in \mathbb{G}_{n+1}} \mathbb{E}[\varepsilon_k^4 | \mathcal{F}_n] < \infty \quad \text{a.s.}$$

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Estimation of the parameter

We estimate $\theta = (a, b, c, d)^t$ by the

Least squares estimator

$$\hat{\theta}_n = \begin{pmatrix} \hat{a}_n \\ \hat{b}_n \\ \hat{c}_n \\ \hat{d}_n \end{pmatrix} = (\mathbf{I}_2 \otimes \mathbf{S}_{n-1}^{-1}) \sum_{k \in \mathbb{T}_{n-1}} \begin{pmatrix} X_{2k} \\ X_k X_{2k} \\ X_{2k+1} \\ X_k X_{2k+1} \end{pmatrix}$$

where

$$\mathbf{S}_n = \sum_{k \in \mathbb{T}_n} \begin{pmatrix} 1 & X_k \\ X_k & X_k^2 \end{pmatrix}.$$

A martingale identity

$$(\hat{\theta}_n - \theta) = (\mathbf{I}_2 \otimes \mathbf{S}_{n-1}^{-1}) \sum_{k \in \mathbb{T}_{n-1}} \begin{pmatrix} \varepsilon_{2k} \\ X_k \varepsilon_{2k} \\ \varepsilon_{2k+1} \\ X_k \varepsilon_{2k+1} \end{pmatrix}.$$

Consequently, if $\Sigma_n = \mathbf{I}_2 \otimes \mathbf{S}_n$, we have the martingale identity

$$\hat{\theta}_n - \theta = \Sigma_{n-1}^{-1} M_n$$

where M_n is the martingale given by

$$M_n = \sum_{k \in \mathbb{T}_{n-1}} \begin{pmatrix} \varepsilon_{2k} \\ X_k \varepsilon_{2k} \\ \varepsilon_{2k+1} \\ X_k \varepsilon_{2k+1} \end{pmatrix}.$$

Estimation of the conditional variance and covariance

We estimate the conditional variance σ^2 and covariance ρ by

Conditional variance estimator

$$\hat{\sigma}_n^2 = \frac{1}{2|\mathbb{T}_{n-1}|} \sum_{k \in \mathbb{T}_{n-1}} (\hat{\varepsilon}_{2k}^2 + \hat{\varepsilon}_{2k+1}^2)$$

Conditional covariance estimator

$$\hat{\rho}_n = \frac{1}{|\mathbb{T}_{n-1}|} \sum_{k \in \mathbb{T}_{n-1}} \hat{\varepsilon}_{2k} \hat{\varepsilon}_{2k+1}$$

where for all $k \in \mathbb{G}_n$

$$\begin{cases} \hat{\varepsilon}_{2k} = X_{2k} - \hat{a}_n - \hat{b}_n X_k, \\ \hat{\varepsilon}_{2k+1} = X_{2k+1} - \hat{c}_n - \hat{d}_n X_k. \end{cases}$$

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 - Central limit theorems

Strong laws for the noise

If (ε_n) satisfies (H.1), (H.2) and (H.3), we have

$$\lim_{n \rightarrow +\infty} \frac{1}{|\mathbb{T}_n|} \sum_{k \in \mathbb{T}_n} \varepsilon_k = 0 \quad \text{a.s.}$$

$$\lim_{n \rightarrow +\infty} \frac{1}{|\mathbb{T}_n|} \sum_{k \in \mathbb{T}_n} \varepsilon_k^2 = \sigma^2 \quad \text{a.s.}$$

$$\lim_{n \rightarrow +\infty} \frac{1}{|\mathbb{T}_{n-1}|} \sum_{k \in \mathbb{T}_{n-1}} \varepsilon_{2k} \varepsilon_{2k+1} = \rho \quad \text{a.s.}$$

Some notations

First of all, let

$$\bar{a} = \frac{a+c}{2}, \quad \bar{b} = \frac{b+d}{2},$$
$$\overline{ab} = \frac{ab+cd}{2}, \quad \overline{a^2} = \frac{a^2+c^2}{2}, \quad \overline{b^2} = \frac{b^2+d^2}{2}.$$

In addition, denote by Γ and L the two symmetric matrices

$$\Gamma = \begin{pmatrix} \sigma^2 & \rho \\ \rho & \sigma^2 \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} 1 & \lambda \\ \lambda & \ell \end{pmatrix}$$

where

$$\lambda = \frac{\bar{a}}{1 - \bar{b}} \quad \text{and} \quad \ell = \frac{\overline{a^2} + \sigma^2 + 2\lambda\overline{ab}}{1 - \overline{b^2}}.$$

A keystone lemma

Lemma

If (ε_n) satisfies (H.1), (H.2) and (H.3), we have

$$\lim_{n \rightarrow \infty} \frac{\mathbf{S}_n}{|\mathbb{T}_n|} = L \quad \text{a.s.}$$

Limiting matrix

The limiting matrix L is positive definite.

Strong law for the parameter

Theorem

Assume that (ε_n) satisfies (H.1), (H.2) and (H.3). Then, $\hat{\theta}_n$ **converges a.s. to θ** with the rate of convergence

$$\| \hat{\theta}_n - \theta \|^2 = \mathcal{O} \left(\frac{\log |\mathbb{T}_{n-1}|}{|\mathbb{T}_{n-1}|} \right) \quad \text{a.s.}$$

In addition, if $\Lambda = I_2 \otimes L$, we have the **quadratic strong law**

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |\mathbb{T}_{k-1}| (\hat{\theta}_k - \theta)^t \Lambda (\hat{\theta}_k - \theta) = 4\sigma^2 \quad \text{a.s.}$$

Tools for the strong laws

The strong laws are related to the random variable

$$V_n = (\hat{\theta}_n - \theta)^t \Sigma_{n-1} (\hat{\theta}_n - \theta)$$

where $\Sigma_n = I_2 \otimes S_n$.

Tools for the strong laws

We have the main decomposition

$$V_{n+1} + \mathcal{A}_n = V_1 + \mathcal{B}_{n+1} + \mathcal{W}_{n+1},$$

where, if $\Delta M_{n+1} = M_{n+1} - M_n$,

$$\mathcal{A}_n = \sum_{k=2}^n M_k^t (\Sigma_{k-1}^{-1} - \Sigma_k^{-1}) M_k,$$

$$\mathcal{B}_{n+1} = 2 \sum_{k=2}^n M_k^t \Sigma_k^{-1} \Delta M_{k+1},$$

$$\mathcal{W}_{n+1} = \sum_{k=2}^n \Delta M_{k+1}^t \Sigma_k^{-1} \Delta M_{k+1}.$$

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Strong law for the conditional variance

Denote

$$\sigma_n^2 = \frac{1}{2|\mathbb{T}_{n-1}|} \sum_{k \in \mathbb{T}_{n-1}} (\varepsilon_{2k}^2 + \varepsilon_{2k+1}^2).$$

Theorem

Assume that (ε_n) satisfies (H.1), (H.2) and (H.3). Then, $\hat{\sigma}_n^2$ converges a.s. to σ^2 ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k \in \mathbb{T}_{n-1}} (\hat{\varepsilon}_{2k} - \varepsilon_{2k})^2 + (\hat{\varepsilon}_{2k+1} - \varepsilon_{2k+1})^2 = 4\sigma^2 \quad \text{a.s.}$$

$$\lim_{n \rightarrow \infty} \frac{|\mathbb{T}_n|}{n} (\hat{\sigma}_n^2 - \sigma_n^2) = 4\sigma^2 \quad \text{a.s.}$$

Strong law for the conditional covariance

Denote

$$\rho_n = \frac{1}{|\mathbb{T}_{n-1}|} \sum_{k \in \mathbb{T}_{n-1}} \varepsilon_{2k} \varepsilon_{2k+1}.$$

Theorem

Assume that (ε_n) satisfies (H.1), (H.2) and (H.3). Then, $\hat{\rho}_n$ converges a.s. to ρ ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k \in \mathbb{T}_{n-1}} (\hat{\varepsilon}_{2k} - \varepsilon_{2k})(\hat{\varepsilon}_{2k+1} - \varepsilon_{2k+1}) = 2\rho \quad \text{a.s.}$$

$$\lim_{n \rightarrow \infty} \frac{|\mathbb{T}_n|}{n} (\hat{\rho}_n - \rho_n) = 4\rho \quad \text{a.s.}$$

More assumptions

(H.4) $\forall n \geq 0$ and $\forall k \in \mathbb{G}_{n+1}$,

$$\mathbb{E}[\varepsilon_k^4 | \mathcal{F}_n] = \tau^4 \quad \text{a.s.}$$

Moreover, $\forall k \neq \ell \in \mathbb{G}_{n+1}$ with $[k/2] = [\ell/2]$ and for $\nu^2 < \tau^4$

$$\mathbb{E}[\varepsilon_k^2 \varepsilon_\ell^2 | \mathcal{F}_n] = \nu^2 \quad \text{a.s.}$$

(H.5)

$$\sup_{n \geq 0} \sup_{k \in \mathbb{G}_{n+1}} \mathbb{E}[\varepsilon_k^8 | \mathcal{F}_n] < \infty \quad \text{a.s.}$$

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Central limit theorems

Theorem

If (ε_n) satisfies (H.1) to (H.5), we have

$$\sqrt{|\mathbb{T}_{n-1}|}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{L}} \mathcal{N}(\mathbf{0}, \Gamma \otimes L^{-1}),$$

$$\sqrt{|\mathbb{T}_{n-1}|}(\hat{\sigma}_n^2 - \sigma^2) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{\tau^4 - 2\sigma^4 + \nu^2}{2}\right),$$

$$\sqrt{|\mathbb{T}_{n-1}|}(\hat{\rho}_n - \rho) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \nu^2 - \rho^2).$$

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Central limit theorems

Theorem

If (ε_n) satisfies (H.1) to (H.5), we have

$$\sqrt{|\mathbb{T}_{n-1}|}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{L}} \mathcal{N}(\mathbf{0}, \Gamma \otimes L^{-1}),$$

$$\sqrt{|\mathbb{T}_{n-1}|}(\hat{\sigma}_n^2 - \sigma^2) \xrightarrow{\mathcal{L}} \mathcal{N}\left(\mathbf{0}, \frac{\tau^4 - 2\sigma^4 + \nu^2}{2}\right),$$

$$\sqrt{|\mathbb{T}_{n-1}|}(\hat{\rho}_n - \rho) \xrightarrow{\mathcal{L}} \mathcal{N}(\mathbf{0}, \nu^2 - \rho^2).$$

!!!! Many thanks for your attention !!!!

