LOCALLY NILPOTENT DERIVATIONS OF RINGS WITH ROOTS ADJOINED

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Abstract. Working over a ground field $\mathbf{k}$ of characteristic zero, we lay out the basic theory for locally nilpotent derivations of rings of the form $B = R[z]$, where R is a commutative $\mathbf{k}$-domain, and $z^n \in R$ for some positive integer n. One goal of doing so is to identify common elements among disparate problems involving locally nilpotent derivations, and to establish a theoretical framework which serves to unify them. In order to study applications, we introduce the *absolute degree* of ring elements, and the *homogenization* of an affine ring and its derivations. General conditions to determine when B is rigid or non-rigid are given. The theory is then applied to a wide range of examples, including $\mathbb{Z}$-graded polynomial rings, the Russell cubic threefold, and Pham-Brieskorn varieties.

1. Introduction

Suppose $\mathbf{k}$ is a field of characteristic zero. This paper investigates locally nilpotent derivations of rings of the form $B = R[z]$, where R is a commutative $\mathbf{k}$-domain, and $z^n \in R$ for some positive integer n. Such a ring has a natural grading by $\mathbb{Z}_n$, and *Theorem 4.1* gives the basic properties of locally nilpotent derivations D of B which are homogeneous relative to this grading. In particular, D is always a quasi-extension of a locally nilpotent derivation δ of R, and $D^2 z = 0$. There are two cases which can occur: (1) $Dz = 0$, in which case $\ker D$ is a free module over $\ker \delta$ of rank n, generated by the powers of z; or (2) $Dz \neq 0$, in which case $\ker D = \ker \delta$, and z^n is a local slice of δ. In the first case, for the quotient maps of the corresponding $\mathbb{G}_a$-actions, a generic orbit of $\mathrm{Spec}(R)$ will divide into n orbits of $\mathrm{Spec}(B)$.

Section 2 gives properties of locally nilpotent derivations, and introduces the *absolute degree* $|f|_R$ of elements of R. In several of our results, it is important to know whether $|f|_R \leq 1$ i.e, if f is in the kernel, or is a local slice, of a non-zero locally nilpotent derivation of R. *Corollary 4.1* shows that, if $z^n = f \in R$, then there is a one-to-one correspondence between elements of $\delta \in \mathrm{LND}(R)$ with $\delta^2 f = 0$ and homogeneous elements of $\mathrm{LND}(B)$. This correspondence is given by choosing the appropriate quasi-extension, and is a tool for studying the locally nilpotent derivations of B by looking at those of R. For example, *Cor. 4.2* shows that if R is $\mathbb{Z}$-graded, f is $\mathbb{Z}$-homogeneous of degree coprime to n, and $|f|_R \geq 2$, then B is rigid, i.e., B has no non-zero locally nilpotent derivations.

The process of homogenizing a polynomial relative to a positive system of weights on $\mathbb{A}^n$ is a classical tool in algebra and algebraic geometry. In Section 3, we generalize this process to affine rings and their derivations. When the ring B has a $\mathbb{Z}$-filtration, the associated $\mathbb{Z}$-graded ring $\mathrm{gr}(B)$ is defined. One way to understand the locally nilpotent derivations of B is to study those of $\mathrm{gr}(B)$. However, this approach can be complicated. As an alternative, we construct a $\mathbb{Z}$-graded ring B^H from B when certain data are given. B^H contains a homogeneous element w such that $B^H/(w-1)B^H = B$; in general, we also have $B^H/wB^H = \mathrm{gr}(B)$. By studying the homogeneous locally nilpotent derivations of B^H, together with the morphism $B^H \to B$, we obtain a promising new approach to understanding the locally nilpotent derivations of B. Geometrically, we consider the algebraic variety $X = \mathrm{Spec}(B)$ with an embedding $\varphi : X \to \mathbb{A}^n$ and a system of integer weights ω on $\mathbb{A}^n$. Any $\mathbb{G}_a$-action ρ on $X = \mathrm{Spec}(B)$ induces a $\mathbb{G}_a$-action ρ^H on $X^H = \mathrm{Spec}(B^H)$, and

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each non-zero fiber of the associated equivariant $\mathbb{A}^1$-fibration of X^H is isomorphic to X. The action ρ^H distinguishes information about ρ in much the same way that the blow-up of a variety unpacks information about its subvarieties. We call ρ^H the *expansion* of ρ relative to given data (X, φ, ω).

Section 3 also features *Thm. 3.1*, which asserts that the degree of a non-zero derivation of an affine **k**-domain B is finite for any degree function inherited by B from a $\mathbb{Z}$-grading on one of its localizations. The framework of homogenizations laid out in the first part of the section allows us to give a short proof of this result. By comparison, Daigle recently discovered pathological degree functions on polynomial rings, with the property that every derivation has infinite degree. See *Remark 3.3* and *Remark 3.4*. As a corollary to *Thm. 3.1*, we establish the basic fact that any **k**-derivation of a $\mathbb{Z}$-graded affine **k**-domain is the sum of finitely many homogeneous derivations (*Cor. 3.1*). This fact is used throughout the paper.

The rest of the paper turns to applications. In Section 6, *Thm. 5.1* is applied to give a proof that the Makar-Limanov invariant of the Russell cubic threefold is non-trivial. In this proof, the associated graded rings used in Makar-Limanov's original proof are reinterpreted by homogenization. In Section 7, *Thm. 4.1* and *Thm. 5.1* are used to establish a fundamental result about homogeneous locally nilpotent derivations of the polynomial ring $A^{[n]}$ over an integral domain A containing $\mathbb{Q}$. *Theorem 7.1* provides surprisingly simple and effective numerical criteria which, when satisfied, imply that certain variables of $A^{[n]}$ are either local slices or invariants for *all* homogeneous locally nilpotent derivations. Thus, from a finite sequence of integers, we can immediately deduce invariant properties of homogeneous derivations which are otherwise difficult to calculate. In addition, this result does not assume that the derivations in question are A-derivations, only that the grading is an A-grading.

In Sections 8 and 9, the theory is applied to Pham-Brieskorn surfaces and related varieties. Recently, several authors have studied the $\mathbb{G}_a$-actions of these varieties, most notably, Kaliman and Zaidenberg [13], Kaliman and Makar-Limanov [12], and Crachiola and Maubach [3]. An important tool in these papers is Mason's Theorem, which gives a bound for the degrees of $f, g, h \in K[t]$ (a univariate polynomial ring over a field K) when $f + g + h = 0$; see [18, 21]. This is due to the fact that, if B is a commutative $\mathbb{Q}$-domain and D is a non-zero locally nilpotent derivation of B, then $B \subset K[t]$, where K is the field of fractions of the kernel of D, and $t \in B$ is a local slice of D. Thus, Mason's Theorem can be applied to elements of B. However, the degree bounds which it yields do not settle all cases, and the bounds become even weaker when generalized to more than 3 terms; see DeBondt [8] for an overview of Mason's Theorem and its generalizations. Our approach combines the new results of Sections 2-5 with established degree bounds, allowing us to settle many new cases, including several of the open cases listed in [3].

The paper is organized as follows.

Preliminaries. We assume throughout that $\mathbf{k}$ is a ground field of characteristic zero, and rings are commutative. The polynomial ring in n variables over a ring B is denoted by $B^{[n]}$. By a *degree function* on a ring B, we mean a function $\deg : B \to \mathbb{Z} \cup \{\infty\}$ such that, for all $a, b \in B$:

1. $\deg b = 0$ if and only if $b = 0$
2. $\deg(ab) = \deg a + \deg b$
3. $\deg(a + b) \leq \max\{\deg a, \deg b\}$

Suppose that the ring B is $\mathbb{Z}$-graded: $B = \oplus_{i \in \mathbb{Z}} B_i$. Given $i \in \mathbb{Z}$, there exists a unique function $h_i : B \to B_i$ such that, for each $f \in B$, $f = \sum_{i \in \mathbb{Z}} h_i(f)$. Given $f \in B$, the image $h_i(f)$ is denoted by f_i. The assignment $f \to \max\{i | f_i \neq 0\}$ defines a degree function on B. A derivation $D \in \mathrm{Der}(B)$ is *homogeneous* if and only if there exists $d \in \mathbb{Z}$ such that $DB_i \subset B_{i+d}$ for all $i \in \mathbb{Z}$. In this case, the *degree* of D equals d. If R is a subring of B, the term B is $\mathbb{Z}$-graded *over* R means $R \subset B_0$. When $\mathbf{k} \subset B$, any $\mathbb{Z}$-grading of B is assumed to be over $\mathbf{k}$.

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2. Locally Nilpotent Derivations

We first recall a few basic definitions and facts concerning locally nilpotent derivations. For a more extensive treatment of the subject, the reader is referred to [11].

Suppose B is a commutative $\mathbf{k}$-domain. The set of $\mathbf{k}$-derivations of B is denoted by $\mathrm{Der}_{\mathbf{k}}(B)$, and the subalgebra of elements at which $D \in \mathrm{Der}_{\mathbf{k}}(B)$ is nilpotent is

$$\mathrm{Nil}(D) = \{f \in B \,|\, D^n f = 0 \,\mathrm{for}\, n \gg 0\} \ .$$

$D \in \mathrm{Der}_{\mathbf{k}}(B)$ is *locally nilpotent* if and only if $\mathrm{Nil}(D) = B$. The set of locally nilpotent derivations of B is denoted by $\mathrm{LND}(B)$. If A is a subring of B, then

$$\mathrm{LND}_A(B) = \{D \in \mathrm{LND}(B) \,|\, DA = 0\} \ .$$

A non-zero element $D \in \mathrm{LND}(B)$ induces a degree function ν_D on B, namely, for non-zero $f \in B$,

$$\nu_D(f) = \max\{n \,|\, D^{n+1} f = 0\} \ .$$

Any $f \in B$ with $\nu_D(f) = 1$ is called a *local slice* for D. The subalgebra

$$ML(B) = \cap_{D \in \mathrm{LND}(B)} \ker D$$

is the *Makar-Limanov invariant* of B. Note that we always have $B^* \subset ML(B)$. B is said to be *rigid* if and only if $ML(B) = B$, or equivalently, $\mathrm{LND}(B) = \{0\}$. And B is *stably rigid* if and only if $\mathrm{LND}(B^{[n]}) = \mathrm{LND}_B(B^{[n]})$ for each integer $n \geq 0$.

2.1. Quasi-extensions.

Let $D : B \to B$ be a derivation of an integral domain B, and let $\delta : R \to R$ be a derivation of a subring $R \subset B$. Then D is a *quasi-extension* of δ if there exists a nonzero $t \in B$ such that $Ds = t \cdot \delta s$ for all $s \in R$. One of the main tools we use to study locally nilpotent derivations for rings graded by $\mathbb{Z}_n$ is the following.

Lemma 2.1. ([11], Lemma 5.38) *Let B be an integral domain containing $\mathbb{Q}$, and let $D : B \to B$ be a derivation which is a quasi-extension of a derivation $\delta : R \to R$ for some subring R. If $D \in \mathrm{LND}(B)$, then $\delta \in \mathrm{LND}(R)$.*

2.2. Absolute Degree.

Definition 2.1. Suppose B is a commutative $\mathbf{k}$-domain. If B is not rigid, then given $f \in B$, the *absolute degree* of f is defined by

$$|f|_B = \min\{\nu_D(f) \,|\, D \in \mathrm{LND}(B), D \neq 0\} \ .$$

If B is rigid, define $|f|_B = -\infty$ if $f = 0$, and $|f|_B = \infty$ otherwise.

It should be noted that this same definition was given recently by Daigle in [4], where he uses the term *LND-degree* in place of absolute degree. Note also that absolute degree is not a degree function in the standard sense, but satisfies the following properties.

1. $|f|_B = -\infty$ if and only if $f = 0$
2. $|f^m|_B = m|f|_B$ for all integers $m \geq 0$
3. $|fg|_B \geq |f|_B + |g|_B$ for all $f, g \in B$.
4. $|f + \kappa|_B = |f|_B$ for all $f \in B$ and $\kappa \in ML(B)$ with $f, f + \kappa \neq 0$.
5. If K is an algebraic extension field of $\mathbf{k}$ and $B_K = K \otimes_{\mathbf{k}} B$, then $|f|_B \geq |f|_{B_K}$.

If B is $\mathbb{Z}$-graded and affine, then the absolute degree satisfies further properties relative to the grading.

6. Given $f \in B$, if $F \in B$ is the highest-degree homogeneous summand of f, then $|f|_B \leq |F|_B$.
7. If $F \in B$ is homogeneous and B is not rigid, then there exists non-zero homogeneous $D \in \text{LND}(B)$ such that
$$|F|_B = \nu_D(F) \ .$$

Definition 2.2. The subalgebra generated by $\{f \in B : |f|_B = 0\}$ is the *Derksen invariant* of B, denoted $\mathcal{D}(B)$.

Note that B is rigid if and only if $\mathcal{D}(B) = \{0\}$.

Lemma 2.2. *Let B be a commutative $\mathbf{k}$-domain.*

(a) *If $\alpha : B \to B$ is a $\mathbf{k}$-algebra automorphism of B, then $\alpha(\mathcal{D}(B)) = \mathcal{D}(B)$.*
(b) *If $\partial \in \text{LND}(B)$, then $\partial(\mathcal{D}(B)) \subset \mathcal{D}(B)$.*

Proof. If B is rigid the result is clear, so assume B is not rigid. Suppose $f \in B$ and $|f|_B = 0$. Then there exists non-zero $D \in \text{LND}(B)$ with $Df = 0$. Since $D' := \alpha D \alpha^{-1} \in \text{LND}(B)$ satisfies $D'(\alpha(f)) = 0$, it follows that $\alpha(f) \in \mathcal{D}(B)$. This suffices to prove part (a), since such elements f generate $\mathcal{D}(B)$ as a $\mathbf{k}$-algebra.

For part (b), note that $\exp(t\partial)$ ($t \in \mathbf{k}$) is a $\mathbb{G}_a$-action on B which, by part (a), restricts to a $\mathbb{G}_a$-action on $\mathcal{D}(B)$. Therefore, ∂ restricts to locally nilpotent derivation of $\mathcal{D}(B)$. □

The following well-known result is due to Davenport, dating to 1965.

Theorem 2.1. ([7]) *Let non-zero $u, v \in \mathbf{k}$ and $f, g \in \mathbf{k}[t] = \mathbf{k}^{[1]}$ be given, where f and g are not both constant, together with positive integers l and m. Then relative to standard degrees in t,*

$$\deg(uf^l - vg^m) \geq \frac{1}{m}(lm - l - m) \deg f + 1$$

unless $uf^l = vg^m$ identically.[1]

We use Davenport's Theorem to prove the following result. Part (a) was first given in [17], Lemma 2, whereas part (b) is new.

Theorem 2.2. *Let B be a commutative $\mathbf{k}$-domain, and let $D \in \text{LND}(B)$ be non-zero. Suppose $u, v \in \ker D$ and $x, y \in B$ are non-zero, and a and b are integers with $a, b \geq 2$. Assume $ux^a + vy^b \neq 0$.*

(a) *If $D(ux^a + vy^b) = 0$, then $Dx = Dy = 0$.*
(b) *If $D^2(ux^a + vy^b) = 0$ and a and b are not both 2, then $Dx = Dy = 0$.*

[1]The condition *f and g are not both constant* is missing from Davenport's original formulation, but is necessary for the result to be valid

Proof. Recall that every element of B may be viewed as a univariate polynomial over the field $K = \mathrm{frac}(\ker D)$, since the localization of B at the non-zero elements of $\ker D$ equals $K[t]$, where t is a local slice of D. In this setting, the degree of $f \in B$ equals $\nu_D(f)$, and elements of B of degree 0 are precisely the non-zero elements of $\ker D$.

Since $a, b \geq 2$, we have $ab - a - b \geq 0$. If $Dx \neq 0$ or $Dy \neq 0$, then Davenport's Theorem implies

$$\nu_D(ux^a + vy^b) \geq \frac{1}{a}(ab - a - b)\nu_D(x) + 1 \geq 1 \ , \tag{1}$$

so $D(ux^a + vy^b) \neq 0$. This proves part (a).

For part (b), assume that $a \geq 3$ or $b \geq 3$, and $D^2(ux^a + vy^b) = 0$. Then $ab - a - b \geq 1$. If $D(ux^a + vy^b) \neq 0$, then Dx and Dy are not both 0, and the inequality (1) yields $\nu_D(ux^a + vy^b) > 1$. But then $\nu_D(ux^a + vy^b) \geq 2$, meaning $D^2(ux^a + vy^b) \neq 0$, a contradiction. Therefore, $D(ux^a + vy^b) = 0$. By part (a), $Dx = Dy = 0$. □

Corollary 2.1. *Let $B = \mathbf{k}[x, y] = \mathbf{k}^{[2]}$. If a and b are integers with $a, b \geq 2$, and $a \neq 2$ or $b \neq 2$, then $|x^a + y^b|_B \geq 2$. Consequently, $D^2(x^a + y^b) \neq 0$ for all non-zero $D \in \mathrm{LND}(B)$.*

We note that part (b) of *Thm. 2.2* can also be proved from the following general fact.

Theorem 2.3. *Let B be a commutative $\mathbf{k}$-domain, $D \in \mathrm{LND}(B)$, and $A = \ker D$. Suppose non-zero $f, g \in B$ and $P \in A[X, Y] = A^{[2]}$ are given. If $P(f, g)$ is a local slice of D, then $\nu_D(f)$ divides $\nu_D(g)$ or $\nu_D(g)$ divides $\nu_D(f)$.*

Proof. Let $K = \mathrm{frac}(A)$, let $t = P(f, g)$ be a local slice of B, and let $S = A - \{0\}$. Then $S^{-1}B = K[t]$, and the degree of $b \in B$ relative to t equals $\nu_D(b)$. Since $K[f, g] = K[t]$, it follows from the Epimorphism Theorem [1, 22] that either $\deg f$ divides $\deg g$, or $\deg g$ divides $\deg f$. □

Recently, Daigle proved the following result for polynomials in three variables.

Theorem 2.4. ([4], Prop. 4.7) *Let $B = \mathbf{k}[x, y, z] = \mathbf{k}^{[3]}$. Suppose $a, b, c \geq 2$ and at most one of a, b, c equals 2. Then $|x^a + y^b + z^c|_B \geq 2$. Consequently, $D^2(x^a + y^b + z^c) \neq 0$ for all non-zero $D \in \mathrm{LND}(B)$.*

3. Homogenizations and Degrees

3.1. **Homogenizing an Affine Ring.** Given a positive integer n, let $Q = \mathbf{k}^{[n]}$, and suppose $I \subset Q$ is an ideal. Assume that $\mathfrak{g}$ is a $\mathbb{Z}$-grading of Q, and let deg be the associated degree function on Q. We say that $\mathfrak{g}$ is a *linear* $\mathbb{Z}$-grading of Q if there exist homogeneous $x_1, ..., x_n \in Q$ such that $Q = \mathbf{k}[x_1, ..., x_n]$.

Assume that $\mathfrak{g}$ is a linear $\mathbb{Z}$-grading of Q. For indeterminate w over Q, extend $\mathfrak{g}$ to a $\mathbb{Z}$-grading of $Q[w, w^{-1}]$ by setting $\deg w = 1$, where w is homogeneous. Define a $\mathbf{k}$-automorphism α of $Q[w, w^{-1}]$ by

$$\alpha(x_i) = w^{-\deg x_i} x_i \quad (1 \leq i \leq n) \ , \quad \text{and} \quad \alpha(w) = w \ .$$

Definition 3.1. Let an element $f \in Q$ and an ideal $I \subset Q$ be given.

1. The *homogenization* of f relative to $\mathfrak{g}$ is the element $f^H \in Q[w]$ defined by
$$f^H = w^{\deg f} \alpha(f) \ .$$
2. The *homogenization* of I relative to $\mathfrak{g}$ is the ideal I^H of $Q[w]$ defined by
$$I^H = \langle f^H \mid f \in I \rangle \ .$$

Note that for each $f \in Q$, we have

$$f^H \text{ is homogeneous} \ , \quad \deg(f^H) = \deg(f) \ , \quad \text{and} \quad f^H \notin wQ[w] \ .$$

Definition 3.2. Let the triple $(Q, \mathfrak{g}, I)$ be given, where $Q = \mathbf{k}^{[n]}$ for a positive integer n, $\mathfrak{g}$ is a linear $\mathbb{Z}$-grading of Q, and $I \subset Q$ is an ideal. Let $I^H \subset Q[w]$ be the homogenization of I relative to $\mathfrak{g}$. The $\mathbb{Z}$-graded ring $\mathcal{B}(Q, \mathfrak{g}, I)$ associated to this triple is

$$\mathcal{B}(Q, \mathfrak{g}, I) = Q[w]/I^H ,$$

called the *homogenization* of B relative to $(Q, \mathfrak{g}, I)$. If $B = Q/I$, then B^H will denote $\mathcal{B}(Q, \mathfrak{g}, I)$.

Remark 3.1. In the above definition, if I is a homogeneous ideal relative to $\mathfrak{g}$, then $B = Q/I$ is already $\mathbb{Z}$-graded, and we have

$$I^H = I \cdot Q[w] \qquad \text{and} \qquad B^H = B[w] .$$

Remark 3.2. Let $\mathcal{Q} = Q[w, w^{-1}]$, and observe that

$$\alpha(I \cdot \mathcal{Q}) = \alpha(I) \cdot \mathcal{Q} = I^H \cdot \mathcal{Q} .$$

It follows that α induces the isomorphism

$$\bar{\alpha} : B[w, w^{-1}] = \mathcal{Q}/I \cdot \mathcal{Q} \to \mathcal{Q}/\alpha(I \cdot \mathcal{Q}) = \mathcal{Q}/I^H \cdot \mathcal{Q} = B^H[w^{-1}] .$$

3.2. **Homogenizing a Derivation.**

Definition 3.3. Let the triple $(Q, \mathfrak{g}, I)$ be given as above, and let $D \in \mathrm{Der}_{\mathbf{k}}(B)$ be given, where $B = Q/I$ and $B^H = \mathcal{B}(Q, \mathfrak{g}, I)$. The *homogenization* of D relative to $\mathfrak{g}$ is the homogeneous derivation $D^H \in \mathrm{Der}_{\mathbf{k}}(B^H)$ defined as follows. Extend D to $B[w, w^{-1}]$ by setting $Dw = 0$. Then $\bar{\alpha} D \bar{\alpha}^{-1} \in \mathrm{Der}_{\mathbf{k}}(B^H[w^{-1}])$. Since B^H is affine, there exists a least integer N such that $D^H := w^N \cdot \bar{\alpha} D \bar{\alpha}^{-1}$ restricts to B^H.

Note that if $D \in \mathrm{LND}(B)$, then $D^H \in \mathrm{LND}(B^H)$.

Lemma 3.1. *Let $\pi : B^H \to B$ be the evaluation map $w = 1$. Then $\pi(\ker D^H) = \ker D$.*

Proof. Consider the sequence of maps

$$B \xrightarrow{\iota} B[w, w^{-1}] \xrightarrow{\bar{\alpha}} B^H[w^{-1}] \xrightarrow{\pi} B ,$$

where ι is inclusion. Then $\pi\bar{\alpha}\iota(f) = f$ for all $f \in B$.

Since $w - 1$ is in the kernel of D^H, we have $\pi(\ker D^H) \subset \ker D$. Conversely, given $f \in \ker D$, choose a positive integer m such that $w^m \bar{\alpha}(f) \in B^H$. Then

$$D^H(w^m \bar{\alpha}(f)) = w^N \bar{\alpha} D \bar{\alpha}^{-1}(w^m \bar{\alpha}(f)) = w^{N+m} \bar{\alpha} D(f) = 0 ,$$

i.e., $w^m \bar{\alpha}(f) \in \ker D^H$ and $\pi(w^m \bar{\alpha}(f)) = f$. Therefore, π takes $\ker D^H$ onto $\ker D$. □

3.3. **The Expansion of a $\mathbb{G}_a$-Action.** The construction of the ring B^H and derivation D^H can be described geometrically as follows. Recall that locally nilpotent derivations of B correspond to $\mathbb{G}_a$-actions on $X = \mathrm{Spec}\, B$ via the exponential map.

Suppose that the affine space $\mathbb{A}^n$ is endowed with a linear system of weights ω. Then the weighted projective space $\mathbb{P}^n_\omega$ is the completion of $\mathbb{A}^n$ relative to ω. Any variety $X \subset \mathbb{A}^n$ completes to a subvariety $\overline{X} \subset \mathbb{P}^n_\omega$. In addition, any $\mathbb{G}_a$-action on X extends to the associated affine cone $X^H := \mathcal{C}(\overline{X}) \subset \mathbb{A}^{n+1}$ over $\overline{X}$. At the level of quotients, we have

$$\mathcal{C}(\overline{X})/\!\!/\mathbb{G}_a = \mathcal{C}\left(\overline{X/\!\!/\mathbb{G}_a}\right) .$$

The inclusion $\mathbf{k}[w] \to B^H$ defines an equivariant $\mathbb{A}^1$-fibration $\phi : X^H \to \mathbb{A}^1$, where $\phi^{-1}(\lambda) \cong X$ if $\lambda \in \mathbf{k}^*$, and $\phi^{-1}(0) = \mathrm{gr}(X)$. Here, $\mathrm{gr}(X)$ denotes $\mathrm{Spec}\,(\mathrm{gr}(B))$.

Definition 3.4. Let the triple $(Q, \mathfrak{g}, I)$ be given, where $Q = \mathbf{k}^{[n]}$ for a positive integer n, $\mathfrak{g}$ is a linear $\mathbb{Z}$-grading of Q, and $I \subset Q$ is an ideal. Set $X = \mathrm{Spec}(B)$ and $X^H = \mathrm{Spec}(B^H)$, where $B = Q/I$ and $B^H = \mathcal{B}(Q, \mathfrak{g}, I)$. Given a $\mathbb{G}_a$-action ρ on X, the induced $\mathbb{G}_a$-action on X^H is denoted by ρ^H, where $\rho^H = \exp(tD^H)$ if $\rho = \exp(tD)$ for $D \in \mathrm{LND}(B)$ and $t \in \mathbf{k}$. The $\mathbb{G}_a$-action ρ^H on X^H is the *expansion* of the given $\mathbb{G}_a$-action ρ relative to the triple $(Q, \mathfrak{g}, I)$.

Example. Consider $Q = \mathbf{k}[x,y] = \mathbf{k}^{[2]}$, and let $X \subset \mathbb{A}^2$ be the curve defined by the polynomial $f = y + x^2$. Then X has an essentially unique non-trivial $\mathbb{G}_a$-action ρ, and the action of $t \in \mathbb{G}_a$ on $(x,y) \in X$ is defined by

$$t \cdot (x,y) = (x+t, y - 2tx - t^2) \ .$$

Note that the ring of invariants of ρ is $\mathbf{k}[X]^{\mathbb{G}_a} = \mathbf{k}$, and that ρ is fixed-point free.

Suppose $\mathfrak{g}_1$ is the standard $\mathbb{Z}$-grading of Q, i.e., $\deg x = \deg y = 1$. Then $f^H = wy + x^2$ in $\mathbf{k}[w,x,y]$ and $B^H = \mathbf{k}[w,x,y]/(f^H)$. The completion $\overline{X}$ is a conic in the standard projective plane $\mathbb{P}^2$, and the cone $X_1^H = \mathcal{C}(\overline{X}) \subset \mathbb{A}^3$ is defined by $f^H = wy + x^2 = 0$. The action ρ^H on X_1^H is defined by

$$t \cdot (w,x,y) = (w, x + tw, y - 2tx - t^2 w) \ ,$$

and $\mathbf{k}[X^H]^{\mathbb{G}_a} = \mathbf{k}[w]$. One-dimensional orbits of ρ^H are defined by the ideals $(w - \lambda)B^H$ as λ varies over $\mathbf{k}^*$, and the set of fixed points is the line $w = 0$.

Consider the alternate grading $\mathfrak{g}_2$ of Q defined by $\deg x = 1$ and $\deg y = 2$. Then f is homogeneous, $f^H = f$, and $B^H = B[w]$. In this case, the completion $\overline{X}$ is a line in the weighted projective plane $\mathbb{P}^2_\omega$ for weights $\omega = (1,2)$, and the cone $X_2^H = \mathcal{C}(\overline{X}) \subset \mathbb{A}^3$ is isomorphic to $X \times \mathbb{A}^1 = \mathbb{A}^2$, defined by $y + x^2 = 0$ in $\mathbb{A}^3$. The action ρ^H on X_2^H is defined by

$$t \cdot (w,x,y) = (w, x+t, y - 2tx - t^2) \ .$$

This action is fixed-point free, with orbits defined by $w = \lambda$ as λ varies over $\mathbf{k}$.

The expanded $\mathbb{G}_a$-actions for the two planar $\mathbb{Z}$-gradings $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are depicted in *Fig. 1.*

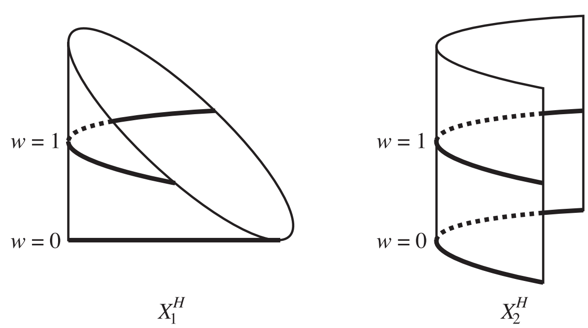

FIGURE 1. Expanded $\mathbb{G}_a$-actions for two gradings of $\mathbb{A}^2$

3.4. **Degrees and Defect.** Assume that B is an affine $\mathbf{k}$-domain with a degree function deg. Given non-zero $D \in \mathrm{Der}_{\mathbf{k}}(B)$, the associated *defect function* def measures the jump in degree which occurs after applying D to an element of B. Specifically, def : $B \to \mathbb{Z} \cup \{-\infty\}$ is defined by

$$\mathrm{def}(b) = \begin{cases} \deg(Db) - \deg(b) & b \neq 0 \\ -\infty & b = 0 \end{cases} .$$

The *degree* of D relative to deg is then defined by $\deg D = \sup_{b \in B} \mathrm{def}(b)$. See Sect. 2.4 of [11] for details about degrees and defect.

The main result needed is the following, which generalizes results of [19, 23].

Theorem 3.1. *Let $B \subset B'$ be affine $\mathbf{k}$-algebras, where B' is a $\mathbb{Z}$-graded localization of B. Let* deg *be the induced degree function of B' restricted to B. Then $\deg D < \infty$ for all $D \in \mathrm{Der}_{\mathbf{k}}(B)$.*

Proof. Assume first that B is $\mathbb{Z}$-graded, and write $B = \mathbf{k}[y_1, ..., y_n]$, where each y_i is homogeneous. We may assume $D \neq 0$. In this case, if $m = \max_{1 \le i \le n} \mathrm{def}(y_i)$, then $m \in \mathbb{Z}$.

Define $Q = \mathbf{k}[x_1, ..., x_n] = \mathbf{k}^{[n]}$ and let $p : Q \to B$ be the surjection defined by $p(x_i) = y_i$. Then $B = Q/I$, where Q has a linear $\mathbb{Z}$-grading $\mathfrak{g}$ defined by setting $\deg x_i = \deg y_i$, and $I = \ker p$ is a homogeneous ideal of Q. The triple $(Q, \mathfrak{g}, I)$ defines the ring $B^H = B[w]$, and the automorphism $\bar{\alpha}$ of $B[w, w^{-1}]$ (defined as in *Remark 3.2* above) is given by

$$\bar{\alpha}(y_i) = w^{-\deg y_i} y_i \quad (1 \le i \le n) \quad \text{and} \quad \bar{\alpha}(w) = w \ .$$

Given $f \in B$, define $f^H \in B^H$ by $f^H = w^{\deg f} \bar{\alpha}(f)$. We have:

$$\bar{\alpha} D \bar{\alpha}^{-1}(f^H) = \begin{cases} w^{-\mathrm{def}(f)} (Df)^H & Df \neq 0 \\ 0 & Df = 0 \end{cases}$$

It follows that, for all $f \in B$ with $Df \neq 0$,

$$D^H f^H = w^m \cdot \bar{\alpha} D \bar{\alpha}^{-1}(f^H) = w^{m - \mathrm{def}(f)} (Df)^H \in B[w] \ . \tag{2}$$

Since $(Df)^H \notin wB[w]$, we conclude that $m - \mathrm{def}(f) \ge 0$. Therefore, $m \ge \mathrm{def}(g)$ for all $g \in B$. This proves that $\deg D \le m < \infty$ when B is $\mathbb{Z}$-graded.

Now suppose that B' is a $\mathbb{Z}$-graded localization of B. Since B' is a localization of B, D extends to a derivation D' on B' via the quotient rule. By what was shown above, if $\deg'$ is the degree function on B' induced by the $\mathbb{Z}$-grading, then $\deg' D' < \infty$. Since D is the restriction of D' to B, and deg is the restriction of $\deg'$ to B, it follows that $\deg D < \infty$. □

Corollary 3.1. *Suppose that B is a $\mathbb{Z}$-graded affine* **k***-domain. Given $D \in \mathrm{Der}_{\mathbf{k}}(B)$, there exist $m, n \in \mathbb{Z}$ and homogeneous $D_i \in \mathrm{Der}_{\mathbf{k}}(B)$ ($i \in \mathbb{Z}$, $m \le i \le n$) such that $\deg D_i = i$ and $D = \sum_{i=m}^{n} D_i$.*

Proof. Given $i, j \in \mathbb{Z}$, define $D_i : B_j \to B_{i+j}$ by $D_i(f) = (Df)_{i+j}$, and extend D_i linearly to all B. By definition, D_i is homogeneous for each i, and $D = \sum_{i \in \mathbb{Z}} D_i$. Moreover, the reader can easily verify that $D_i \in \mathrm{Der}_{\mathbf{k}}(B)$ for each i. It must be shown that $D_N = 0$ for $|N| \gg 0$.

By the preceding theorem, D respects the $\mathbb{Z}$-filtration induced by the $\mathbb{Z}$-grading of B, i.e., there exists $d \in \mathbb{Z}$ such that, if $A_j = \oplus_{k \le j} B_k$ ($j \in \mathbb{Z}$), then $D(A_j) \subset A_{j+d}$ for each $j \in \mathbb{Z}$. In particular, given $N \in \mathbb{Z}$, it follows that for all $j \in \mathbb{Z}$,

$$D_N(B_j) \subset A_{j+d} \cap B_{j+N} = \{0\} \quad \text{for} \quad N > d \ .$$

Therefore, $D_N = 0$ for $N > d$.

The fact that $D_N = 0$ for $N \ll 0$ follows by considering the alternate grading of B defined by $B = \oplus_{i \in \mathbb{Z}} B'_i$, where $B'_i = B_{-i}$. This completes the proof. □

Definition 3.5. In the notation of the corollary, the derivation D_n is called the *highest-degree homogeneous summand* of D relative to the given $\mathbb{Z}$-grading of B.

Note that if the affine domain B is $\mathbb{Z}$-graded and $D \in \mathrm{LND}(B)$, then the highest-degree homogeneous summand D_N is locally nilpotent, and if $f \in \ker D$ has highest-degree homogeneous summand f_k, then $D_N(f_k) = 0$. See [11], Princ. 14.

Corollary 3.2. *Suppose that $Q = \mathbf{k}^{[n]}$ with linear $\mathbb{Z}$-grading $\mathfrak{g}$, and let $I \subset Q$ be a homogeneous ideal. Then $B = Q/I$ is $\mathbb{Z}$-graded, and $B = \mathbf{k}[y_1, ..., y_n]$ for homogeneous y_i. Relative to the triple $(Q, \mathfrak{g}, I)$, the following hold.*

- **(a)** *If $f \in B$ is non-zero and $f = \sum_{i \le k} f_i$ is its decomposition into homogeneous summands, then $f_k \neq 0$ and*
$$f^H = \sum_{i \le k} w^{k-i} f_i \ .$$
- **(b)** *If $D \in \mathrm{Der}_{\mathbf{k}}(B)$ is non-zero and $D = \sum_{i \le N} D_i$ is its decomposition into homogeneous summands, then for each j ($1 \le j \le n$),*
$$D^H(y_j) = w^{m - \mathrm{def}(y_j)} \Delta(y_j) \ , \quad \text{where} \quad \Delta = \sum_{i \le N} w^{N-i} D_i \quad \text{and} \quad m = \max_i \mathrm{def}(y_i) \ .$$

Proof. Consider first the degree-i homogeneous summand f_i of f. By hypothesis, there exists homogeneous $P \in Q = \mathbf{k}[x_1, ..., x_n]$ such that $f_i = P(y_1, ..., y_n)$. Therefore,

$$\bar{\alpha}(f_i) = P(\bar{\alpha}(y_1), ..., \bar{\alpha}(y_n)) = P(w^{-\deg y_1} y_1, ..., w^{-\deg y_n} y_n) = w^{-\deg f_i} P(y_1, ..., y_n) = w^{-i} f_i \ .$$

It follows that

$$f^H = w^{\deg f} \bar{\alpha}(f) = w^n \bar{\alpha}\left(\sum f_i\right) = \sum w^n \bar{\alpha}(f_i) = \sum w^{n-i} f_i \ .$$

For part (b), equation (2) above shows that

$$D^H(y_j) = w^{m-\mathrm{def}(y_j)} (Dy_j)^H = w^{m-\mathrm{def}(y_j)} \left(\sum_{i \le N} D_i y_j \right)^H \ .$$

From part (a), it follows that

$$D^H(y_j) = w^{m-\mathrm{def}(y_j)} \sum_{i \le N} w^{(N+j)-(i+j)} D_i y_j = w^{m-\mathrm{def}(y_j)} \left(\sum_{i \le N} w^{N-i} D_i \right) (y_j) \ .$$

□

Recall that an ideal I of the $\mathbf{k}$-algebra B is an *integral ideal* for $D \in \mathrm{Der}_{\mathbf{k}}(B)$ if $DI \subset I$.

Corollary 3.3. *For $n \ge 1$, let $Q = \mathbf{k}^{[n]}$ with linear $\mathbb{Z}$-grading $\mathfrak{g}$. Given $D \in \mathrm{Der}_{\mathbf{k}}(Q)$, if $I \subset Q$ is an integral ideal of D, then $I^H \subset Q^H = Q[w]$ is an integral ideal of D^H.*

Proof. By hypothesis, $DI \subset I$. Let $g \in I$ be given. If $Dg = 0$, then $g^H = 0$ implies $D^H(g^H) = 0 \in I^H$. So assume $Dg \ne 0$. From equation (2) we have

$$D^H(g^H) = w^{m-\mathrm{def}(g)} (Dg)^H \ ,$$

where $m - \mathrm{def}(g) \ge 0$. Since $Dg \in I$, it follows that $(Dg)^H \in I^H$. Therefore $D^H(g^H) \in I^H$ for all $g \in I$.

Now suppose that $p \in I^H$ is given. By definition, there exist $a_1, ..., a_r \in I$ and $c_1, ..., c_r \in Q^H$ such that $p = \sum_{1 \le i \le r} c_i a_i^H$. Therefore,

$$D^H p = \sum_{1 \le i \le r} c_i D^H(a_i^H) + a_i^H D^H(c_i) \ .$$

Since each a_i^H and $D^H(a_i^H)$ belongs to I^H, it follows that $D^H p \in I^H$. Therefore, $D^H(I^H) \subset I^H$. □

Remark 3.3. In [23], Cor.2.2.7, Wang showed *Thm. 3.1* in the case B is $\mathbb{Z}$-graded and D is locally nilpotent. Then Nur [19] extended Wang's result to all derivations D of B. Daigle proved the theorem in the case that the degree function on B is induced by a locally nilpotent derivation, which is a special case of the condition that deg arises as the restriction of the degree function on a $\mathbb{Z}$-graded localization B; see [11], Thm. 2.11. Finally, in the recent preprint [5], Prop. 4.6, Daigle generalizes *Thm. 3.1* by replacing $\mathbb{Z}$ with a totally ordered abelian group.

Remark 3.4. The conclusion of *Thm. 3.1* is false for general degree functions. In the recent preprint [5], Daigle defines a degree function on $B = \mathbf{k}^{[2]}$ with the property that $\deg D = \infty$ for all non-zero derivations D of B.

Remark 3.5. The process of homogenization described in this section uses a polynomial ring Q over the ground field $\mathbf{k}$. However, this process and most of the subsequent results can be generalized to polynomial rings $Q = A^{[n]}$, where A is a $\mathbf{k}$-domain, and the $\mathbb{Z}$-grading of Q is assumed to be over A. The arguments go through almost unchanged.

4. Adjoining One Root

In this section, we assume the following.

1. R is a $\mathbf{k}$-domain
2. $f \in R$ and $n \in \mathbb{Z}$, $n \geq 2$
3. $R[z] = R^{[1]}$ and $f + z^n$ is prime

Define

$$B = R[z]/(f + z^n) \ .$$

We immediately observe:

Lemma 4.1. *If $|f|_R \leq 1$, then B is not rigid.*

Proof. By hypothesis, $f \neq 0$ and there exists non-zero $\delta \in \mathrm{LND}(R)$ such that $\delta^2 f = 0$. Define $D \in \mathrm{Der}_{\mathbf{k}}(B)$ by

$$Dr = z^{n-1}\delta r \quad (r \in R) \quad \text{and} \quad Dz = -\tfrac{1}{n}\delta f \ .$$

Then D is non-zero. It must be shown that D is locally nilpotent.

Note first that, given $r \in R$, $\delta r = 0$ implies $Dr = 0$. Therefore, $\ker \delta \subset \mathrm{Nil}(D)$. Since $\delta f \in \ker \delta$, it follows that $Dz \in \ker D$, which implies $z \in \mathrm{Nil}(D)$. Since R and z generate B, it will therefore suffice to show that $R \subset \mathrm{Nil}(D)$. This is shown by induction on $N = \nu_\delta(r)$ for $r \in R$. The basis for induction has been established, since $\ker \delta \subset \mathrm{Nil}(D)$.

Given $N \geq 1$, assume $s \in \mathrm{Nil}(D)$ whenever $\nu_\delta(s) \leq N - 1$. In particular, if $r \in R$ is such that $\nu_\delta(r) = N$, then $\delta r \in \mathrm{Nil}(D)$, since $\nu_\delta(\delta r) = N - 1$. It follows that

$$Dr = z^{n-1}\delta r \in \mathrm{Nil}(D) \quad \Rightarrow \quad r \in \mathrm{Nil}(D) \ .$$

Therefore, $\mathrm{Nil}(D) = B$, i.e., D is locally nilpotent. □

Remark 4.1. The converse of this lemma may fail to hold. For example, let $f = x^2 + y^3$ in $\mathbb{C}[x, y] = \mathbb{C}^{[2]}$. Then the results of the preceding section show $|f|_R = 2$, but the ring

$$B = R[z]/(f + z^2) = \mathbb{C}[x, y, z]/(x^2 + y^3 + z^2)$$

is the coordinate ring of a non-rigid Pham-Brieskorn surface (see *Sect. 8* below). However, under certain additional assumptions, the condition $|f|_R \geq 2$ becomes equivalent to the condition B is rigid; see *Cor. 4.2* below.

We next observe that B is a free R-module, given by

$$B = R + Rz + \cdots Rz^{n-1} \ .$$

If $\deg z \in \mathbb{Z}^*$, then this decomposition defines a $\mathbb{Z}_n$-grading of B over R for which z is homogeneous, namely, $B = \oplus_{0 \leq i \leq n-1} B_i$ for $B_i = Rz^i$.

Definition 4.1. If $D \in \mathrm{Der}_{\mathbf{k}}(B)$ is homogeneous relative to the $\mathbb{Z}_n$-grading, $D \neq 0$, then $\lambda(D) \in \mathbb{Z}_n$ is defined by $\lambda(D) \deg z = \deg D$.

Lemma 4.2. *Let $D \in \mathrm{Der}_{\mathbf{k}}(B)$ be non-zero and homogeneous relative to the $\mathbb{Z}_n$-grading. Then there exists $\delta \in \mathrm{Der}_{\mathbf{k}}(R)$ such that D is a quasi-extension of δ, namely, $D|_R = z^\lambda \delta$, where $\lambda = \lambda(D)$. In addition, if $D \in \mathrm{LND}(B)$, then $\delta \in \mathrm{LND}(R)$.*

Proof. Given $s \in R = B_0$, Ds is homogeneous, and therefore $Ds \in Rz^k$ for some k $(0 \leq k \leq n - 1)$. The derivation D is not identically zero on R, and therefore we can choose s such that $Ds \neq 0$. It follows that

$$Ds \in Rz^k \quad \Rightarrow \quad \deg s + \deg D \equiv k \deg z \pmod{n} \quad \Rightarrow \quad \lambda \deg z \equiv k \deg z \pmod{n} \quad \Rightarrow \quad \lambda = k \ .$$

Therefore, $DR \subset Rz^\lambda$. Define $\delta : R \to R$ by $\delta s = z^{-\lambda} Ds$. Then δ is a well-defined $\mathbf{k}$-derivation of R, and D is a quasi-extension of δ. In addition, if D is locally nilpotent, then δ is also locally nilpotent, by *Lemma 2.1* above. □

Remark 4.2. It follows that, for every $D \in \mathrm{Der}_{\mathbf{k}}(B)$, there exist $\delta_i \in \mathrm{Der}_{\mathbf{k}}(R)$, $0 \le i \le n-1$, such that

$$D|_R = \delta_0 + z\delta_1 + z^2\delta_2 + \cdots + z^{n-1}\delta_{n-1} \ .$$

Theorem 4.1. *Suppose that B is $\mathbb{Z}_n$-graded over R, where z is homogeneous and $\deg z \in \mathbb{Z}_n^*$. Given homogeneous $D \in \mathrm{LND}(B)$, let $\delta \in \mathrm{LND}(R)$ be such that $D|_R = z^\lambda \delta$.*

- **(a)** $Dz, \delta f \in \ker \delta = R \cap \ker D$
- **(b)** *If $Dz \neq 0$, then:*
  - **(i)** $\lambda(D) = n-1$
  - **(ii)** $\ker D = \ker \delta$

Proof. Set $\lambda = \lambda(D)$. Since Dz is homogeneous, $Dz \in Rz^k$ for some k ($0 \le k \le n-1$). If $k \neq 0$, then z divides Dz, implying that $Dz = 0$. Otherwise, $k = 0$. So $Dz \in R$ in either case.

If $Dz \neq 0$, then $D(z^n) \neq 0$, since $\ker D$ is algebraically closed in B. Thus,

$$z^n \in R \quad \Rightarrow \quad D(z^n) = nz^{n-1}Dz \in DR \subset Rz^\lambda \quad \Rightarrow \quad \lambda = n-1 \ .$$

This proves part (i) of (b).

To finish (a), we must show $D^2z = \delta^2 f = 0$, which is clear if $Dz = 0$. So assume $Dz \neq 0$. Since $Dz \in R$ and $DR \subset Rz^{n-1}$, it follows that $D^2z \in Rz^{n-1}$. Therefore, z divides D^2z, which implies $D^2z = 0$. In addition, since $D(f + z^n) = 0$, it follows that

$$z^{n-1}\delta f + nz^{n-1}Dz = 0 \quad \Rightarrow \quad \delta f = -nDz \in \ker \delta \ .$$

So part (a) is proved.

To finish (b), let $b \in \ker D$ be homogeneous, and write $b = az^k$, where $a \in R$ and $0 \le k \le n-1$. Then $z^k \in \ker D$, meaning that $k = 0$. Therefore, $b = a \in R$. Since $\ker D$ is generated by homogeneous elements, it follows that $\ker D \subset R$ when $Dz \neq 0$. By part (a), it follows that $\ker D = \ker \delta$ in this case. This proves part (ii) of (b). ☐

Corollary 4.1. *Suppose R is a $\mathbf{k}$-domain. Let $f \in R$ and $n \ge 2$ be given, and assume that $B = R[z]/(f + z^n)$ is a domain. The following are equivalent.*

- **(a)** $|f|_R \le 1$
- **(b)** *There exists non-zero $D \in \mathrm{LND}(B)$ which is homogeneous relative to any $\mathbb{Z}_n$-grading of B over R such that z is homogeneous and $\deg z \in \mathbb{Z}_n^*$.*

Proof. That (b) implies (a) is a consequence of *Thm. 4.1(a)*. Conversely, consider the $\mathbb{Z}_n$-grading of B over R induced by the decomposition $B = R + Rz + \cdots Rz^{n-1}$, where $\deg \in \mathbb{Z}_n^*$. If $|f|_R \le 1$, then the derivation D constructed in the proof of *Lemma 4.1* is homogeneous, and thus satisfies the conditions of part (b). ☐

Definition 4.2. Suppose R is a $\mathbf{k}$-domain with $\mathbb{Z}$-grading $\mathfrak{g}$. Assume that $f \in R$ is homogeneous of degree $\phi \in \mathbb{Z}$, and $n \ge 1$. Then $(n\mathfrak{g}, \phi)$ will denote the $\mathbb{Z}$-grading of $B = R[z]/(f + z^n)$ which restricts to the $n\mathbb{Z}$-grading $n\mathfrak{g}$ on R, and for which $\deg z = \phi$.

Corollary 4.2. *Suppose R is a $\mathbb{Z}$-graded affine $\mathbf{k}$-domain, $f \in R$ is homogeneous ($f \neq 0$), $\deg f \neq 0$, and $n \ge 2$ is an integer relatively prime to $\deg f$. Assume that $B = R[z]/(f + z^n)$ is a domain. The following are equivalent.*

- **(a)** $|f|_R \ge 2$
- **(b)** *B is rigid.*

Proof. That (b) implies (a) is a direct consequence of *Cor. 4.1*.

Conversely, assume B is not rigid. Let $\mathfrak{g}$ be the given $\mathbb{Z}$-grading of R, and let $\phi = \deg f$. Consider the induced $\mathbb{Z}$-grading $\mathfrak{h} = (n\mathfrak{g}, \phi)$ of B. Then $\mathfrak{h}$ induces a $\mathbb{Z}_n$-grading of B over R for which $\phi = \deg z \in \mathbb{Z}_n^*$.

Suppose a non-zero $D \in \mathrm{LND}(B)$ is given, and let Δ be the highest-degree homogeneous summand of D relative to $\mathfrak{h}$. (This is where the assumption that R is affine is used.) Then Δ is also non-zero and homogeneous relative to the $\mathbb{Z}_n$-grading. By *Thm. 4.1*, there exists $\delta \in \mathrm{LND}(R)$ such that Δ is a quasi-extension of δ, and $\delta^2 f = 0$. Therefore, $|f|_R \le 1$. □

Corollary 4.3. *Suppose R is a $\mathbb{Z}$-graded affine $\mathbf{k}$-domain, $f \in R$ is homogeneous, and $n \ge 2$ is an integer not dividing $\deg f$. Set $d = \gcd(n, \deg f)$, and assume that the rings*

$$S = R[u]/(f + u^d) \quad \text{and} \quad B = R[z]/(f + z^n)$$

are domains. The following are equivalent.

(a) $|u|_S \ge 2$

(b) *B is rigid.*

Proof. Set $\phi = \deg f$, and let a, b be integers such that $n = da$ and $\phi = db$. By hypothesis, $a \ge 2$, and $\gcd(a, b) = 1$. We have

$$B = R[z]/(f + z^n) = R[z, u]/(f + u^d, u - z^a) = \left(R[u]/(f + u^d)\right)[z]/(u - z^a) = S[z]/(u - z^a)\ .$$

The $\mathbb{Z}$-grading of R extends to S by setting $\deg u = b$, where u is homogeneous. The equivalence of (a) and (b) now follows from *Cor. 4.2.* □

Corollary 4.4. *Suppose R is a $\mathbf{k}$-domain, $\mathfrak{g}$ is a $\mathbb{Z}$-grading of R, and $f \in R$ is homogeneous of degree $\phi \in \mathbb{Z}$ ($\phi \ne 0$). Assume that $n \ge 2$ is an integer such that $B = R[z]/(f + z^n)$ is a domain, and let $\mathfrak{h}$ be the $\mathbb{Z}$-grading of B defined by $\mathfrak{h} = (n\mathfrak{g}, \phi)$. If $\gcd(\phi, n) = 1$, then $D(ML(R)) = 0$ for every homogeneous $D \in \mathrm{LND}(B)$.*

Proof. The $\mathbb{Z}$-grading $\mathfrak{h}$ induces a $\mathbb{Z}_n$-grading of B over R for which $\deg z \in \mathbb{Z}_n^*$, and D is homogeneous relative to the $\mathbb{Z}_n$-grading. By *Thm. 4.1*, D is a quasi-extension of $\delta \in \mathrm{LND}(R)$. Since $\delta(ML(R)) = 0$, it follows that $D(ML(R)) = 0$. □

Corollary 4.5. *Suppose R is an affine k-domain, and $f \in R$ satisfies $|f|_R \ge 2$. Let $m, n \ge \mathbb{Z}$ be such that $m, n \ge 2$, $\gcd(m, n) = 1$, and $f + t^m, f + t^n \in R[t] = R^{[1]}$ are prime. If $R[x, y] = R^{[2]}$, then the ring*

$$B = R[x, y]/(f + x^m y^n)$$

is rigid.

Proof. B is $\mathbb{Z}$-graded over R by setting $\deg x = n$ and $\deg y = -m$. If B_0 is the subalgebra of elements of degree 0, then since $\gcd(m, n) = 1$, we have $B_0 = R[x^m y^n] = R$. It will suffice to show that if $D \in \mathrm{LND}(B)$ is homogeneous, then $D = 0$.

Assume to the contrary that $D \in \mathrm{LND}(B)$ is homogeneous and non-zero.

Consider first the case $\ker D \subset B_0$. Then B_0 is algebraic over $\ker D$, and since $\ker D$ is algebraically closed in B, it follows that $\ker D = B_0 = R$. But then

$$0 = D(f + x^m y^n) = D(x^m y^n) \quad \Rightarrow \quad Dx = Dy = 0\ ,$$

a contradiction since neither x nor y belongs to B_0. So this case cannot occur.

Therefore, there exists non-zero homogeneous $h \in \ker D$ such that $\deg h \ne 0$. If $\deg h > 0$, then h can be expressed as a sum of monomials of the form $r x^\alpha y^\beta$, where $r \in R$ and $\alpha > 0$. Therefore $h \in xB$, which implies $Dx = 0$.

In this case, consider the $\mathbb{Z}_n$-grading of B over R induced by the $\mathbb{Z}$-grading. Since $\deg x = 0$ relative to the $\mathbb{Z}_n$-grading, it follows that $B/(x-1)B$ is $\mathbb{Z}_n$-graded, and that the quotient derivation $\bar{D}$ induced by D is non-zero and $\mathbb{Z}_n$-homogeneous. We have

$$B/(x-1)B = R[x, y]/(f + x^m y^n, x - 1) = R[y]/(f + y^n)\ ,$$

where $\deg y = -m$ is a unit of $\mathbb{Z}_n$. By *Thm. 4.1*, there exists a non-zero $\delta \in \mathrm{LND}(R)$ such that $\delta^2 f = 0$. But this contradicts the hypotheses.

In exactly the same way, a contradiction is reached in the case $\deg h < 0$. We conclude that the only possibility is $\mathrm{LND}(B) = \{0\}$. □

Corollary 4.6. *Suppose R is a rigid affine* k*-domain, and $f \in R$. Let m, n be positive integers such that* $\gcd(m,n) = 1$, *and $f + t^m, f + t^n \in R[t] = R^{[1]}$ are prime. If $R[x,y] = R^{[2]}$, then the ring*

$$B = R[x,y]/(f + x^m y^n)$$

is rigid.

Proof. If $m, n \geq 2$, this follows from *Cor. 4.5*, so assume $m = 1$ or $n = 1$.

Consider first the case $m = n = 1$:

$$B = R[x,y]/(f + xy)\ .$$

Define a $\mathbb{Z}$-grading of B over R by $\deg x = -1$ and $\deg y = 1$. If B_0 is the subalgebra of elements of degree 0, then $B_0 = R[xy] = R$. Let $D \in \mathrm{LND}(B)$ be homogeneous. If $Dx = 0$, then D induces a non-zero quotient derivation on the ring

$$B/(x-1)B = R[x,y]/(f + y, x - 1) \cong_{\mathbf{k}} R\ ,$$

contradicting the fact that R is rigid. Therefore, $Dx \neq 0$, and similarly $Dy \neq 0$.

Let $g \in \ker D$ be homogeneous. If $\deg g < 0$, then $x|g$, since any monomial in x, y over R of negative degree must involve x. But then $Dx = 0$, a contradiction. Similarly, a contradiction is reached if $\deg g > 0$. Therefore, $\deg g = 0$.

So if we assume $D \neq 0$, it follows that $\ker D \subset R$. However, R is algebraic over $\ker D$, and $\ker D$ is algebraically closed in B. Therefore, $\ker D = R$. In particular, $Df = 0$. But then $D(xy) = 0$ implies $Dx = Dy = 0$, and thus $D = 0$, a contradiction. Therefore, the only homogeneous element of $\mathrm{LND}(B)$ is $D = 0$. It follows that B is rigid when $m = n = 1$.

For the remaining cases, we may assume $n = 1$ and $m \geq 2$. Then

$$B = R[x,y]/(f + x^m y) = R[x,y,z]/(f + yz, z - x^m) = \Omega[x]/(z - x^m)\ ,$$

where $\Omega = R[y,z]/(f + yz)$. By what was shown above, Ω is rigid.

Define a $\mathbb{Z}$-grading on Ω by $\deg y = -1$ and $\deg z = 1$. Then $z \in \Omega$ is homogeneous and its degree is relatively prime to $m \geq 2$. By *Cor. 4.2*, it follows that B is rigid. □

Remark 4.3. *Corollary 4.1* admits the following geometric interpretation when the underlying ring R is affine. In this case, when the conditions of the corollary are satisfied, f is either a kernel element or a local slice for D. Consider the affine variety $X = \mathrm{Spec}(B)$: It is endowed with an action of the cyclic group C_n of order n, and the quotient of this action is $Y = \mathrm{Spec}(R)$. Let $\pi_n : X \to Y$ be the quotient map. It is totally ramified over the zero set of f. The homogeneous locally nilpotent derivation D of B induces a $\mathbb{G}_a$-action on X which semi-commutes with the action of C_n. That is, $\mathbb{G}_a \rtimes C_n$ acts on X. This action induces an action of C_n on the algebraic quotient $X/\!\!/\mathbb{G}_a$. Suppose that f is not in the kernel of D. Then the action of C_n on $X/\!\!/\mathbb{G}_a$ is trivial. Thus, the quotient map induces a morphism $Y \to X/\!\!/\mathbb{G}_a$, which is simply the quotient map of the $\mathbb{G}_a$-action on Y induced by δ (where D is a quasi-extension of δ). Consider a generic $\mathbb{G}_a$-orbit L in X: It is isomorphic to an affine line, and $\pi_n(L)$ is a generic orbit of the action induced by δ on Y. Thus $\pi_n(L)$ corresponds to a ramified covering of an affine line onto an affine line. This is only possible if the map is ramified at exactly one point. Thus a generic orbit of the action of δ intersects the zero set of f at exactly one point. In other words, f is a local slice.

5. Adjoining Two Elements

In this section, we continue the assumption that R is a **k**-domain. In addition, assume:

$f \in R$, and $m, n \geq 2$ are relatively prime integers.

For indeterminates y, z over R, define

$$B = R[y,z]/(f + y^m + z^n)\ .$$

Define the subalgebra $S \subset B$ by

$$S = R[y] \cap R[z] = R[y^m] = R[z^n] \ .$$

Then B is a free S-algebra, given by

$$B = \oplus S y^i z^j \quad (0 \le i \le m-1 \ , \ 0 \le j \le n-1) \ .$$

This decomposition defines a $\mathbb{Z}_{mn}$-grading of B over S, where $\deg y = un$ ($u \in \mathbb{Z}_m^*$) and $\deg z = vm$ ($v \in \mathbb{Z}_n^*$).

Theorem 5.1. *Let $D \in \mathrm{LND}(B)$ be homogeneous relative to the given $\mathbb{Z}_{mn}$-grading of B over S.*

- **(a)** $D^2 y = D^2 z = 0$
- **(b)** $Dy = 0$ *or* $Dz = 0$

Proof. The given $\mathbb{Z}_{mn}$-grading induces a $\mathbb{Z}_n$-grading in which $\deg z = m$, $\deg r = 0$ for each non-zero $r \in R[y]$, and D is homogeneous relative to this induced grading. Using $R[y]$ in place of R in *Thm. 5.1*, it follows that $D^2 z = 0$. By symmetry, $D^2 y = 0$ as well, and (a) is proved.

Assume that $Dy \neq 0$ and $Dz \neq 0$. Given homogeneous $b \in \ker D$, write $b = \sigma y^i z^j$ for $\sigma \in S$, $0 \le i \le m-1$, and $0 \le j \le n-1$. Then $i = j = 0$, since $\ker D$ is factorially closed. Therefore, $\ker D \subset S$. It follows that

$$Dy \in S - \{0\} \quad \Rightarrow \quad \deg D + \deg y = 0 \quad \Rightarrow \quad \deg D + un = 0 \quad \Rightarrow \quad n | \deg D \ .$$

Likewise, $Dz \in S - \{0\}$ implies $m | \deg D$. Therefore, $mn | \deg D$, i.e., $\deg D = 0$ in $\mathbb{Z}_{mn}$. But then the fact that $\deg D + \deg y = 0$ implies $\deg y = 0$, a contradiction. This proves (b). □

6. The Russell Cubic Threefold

Let $Q = \mathbf{k}[x, y, z, t] = \mathbf{k}^{[4]}$ and define $f \in Q$ by $f = x + x^2 y + z^2 + t^3$. The Russell cubic threefold $X \subset \mathbb{A}^4$ is defined by $f = 0$. Note that the Derksen invariant $\mathcal{D}(X)$ contains $\mathbf{k}[x, z, t]$. To see this, define $D_1, D_2 \in \mathrm{LND}(\mathbf{k}[X])$ by

$$D_1 = 2z\partial_y - x^2 \partial_z \quad \text{and} \quad D_2 = 3t^2 \partial_y - x^2 \partial_t \ .$$

Then $\ker D_1 = \mathbf{k}[x, t]$ and $\ker D_2 = \mathbf{k}[x, z]$. In fact, it was shown by Makar-Limanov that the Dersksen invariant is equal to $\mathbf{k}[x, z, t]$. There are several proofs of this result, all using degree functions; see for example [11, 12].

The main goal of this section is to prove this fact using the results of the preceding sections.

Theorem 6.1. $\mathcal{D}(X) = \mathbf{k}[x, z, t]$

In order to understand the locally nilpotent derivations of $\mathbf{k}[X]$, form the triple $(Q, \mathfrak{g}, I)$, where $\mathfrak{g} = (-1, 2, 0, 0)$ and $I = fQ$ noting that $\mathbf{k}[X] = Q/I$. Form the homogenization

$$B := \mathcal{B}(Q, \mathfrak{g}, I) = (\mathbf{k}[X])^H = \mathbf{k}[w, x, y, z, t]/(xw + x^2 y + z^2 + t^3) \ .$$

Then B admits two independent $\mathbb{Z}$-gradings, $\mathfrak{g}_1$ and $\mathfrak{g}_2$, corresponding to degree functions

$$\deg_1(w, x, y, z, t) = (0, 6, -6, 3, 2) \quad \text{and} \quad \deg_2(w, x, y, z, t) = (1, -1, 2, 0, 0) \ .$$

Set $Y = \mathrm{Spec}(B)$, and let $\phi : Y \to \mathbb{A}^1$ be the morphism defined by $w \in B$. If Y_c denotes the fiber over $c \in \mathbf{k}$, then $X = Y_1$. For each $c \in \mathbf{k}$, let $B_c = \mathbf{k}[Y_c]$ and let $\pi_c : B \to B_c$ be the natural surjection. Note that, since $\deg_1(w) = 0$, each B_c inherits a $\mathbb{Z}$-grading from $\mathfrak{g}_1$

Lemma 6.1. $|y|_{B_c} \ge 1$ *for each* $c \in \mathbf{k}$.

Proof. The first goal is to show that if $D \in \mathrm{LND}(B_c)$ is non-zero and homogeneous, then either $\ker D = \mathbf{k}[x, z]$ or $\ker D = \mathbf{k}[x, t]$.

Let $R = \mathbf{k}[x, y]$. Then $f \in R$ defined by $f = cx + x^2 y$ is homogeneous of degree 6, and $B_c = R[z, t]/(f + z^2 + t^3)$. The $\mathbb{Z}$-grading of B_c induces a $\mathbb{Z}_6$-grading of B_c over $S = R[z] \cap R[t]$ relative to which $\deg z = 3$ and $\deg t = 2$. By *Thm. 5.1*, it follows that $D^2 z = D^2 t = 0$, and either $Dz = 0$ or $Dt = 0$.

If $Dz = Dt = 0$, then $Df = 0$ implies $Dx = 0$, since $\ker D$ is factorially closed. But then $D = 0$, a contradiction. Therefore, Dz and Dt are not both 0.

Assume $Dt \neq 0$. Then *Thm. 4.1* implies that $D|_{R[z]} = t^2\delta$ for some homogeneous $\delta \in \text{LND}(R[z])$, and $\delta^2(f+z^2) = \delta^2 f = 0$. Since $f = x(c+xy)$, it follows that either $\delta x = 0$ or $\delta(c+xy) = 0$. In either case, $\delta x = 0$. Since $\mathbf{k}[x,z]$ is an algebraically closed subring of B_c, it follows that $\ker D = \mathbf{k}[x,z]$.

The other case, in which $Dt = 0$ and $Dz \neq 0$, follows an identical line of reasoning to show $\ker D = \mathbf{k}[x,t]$.

Now suppose $\theta \in \text{LND}(B_c)$ is non-zero, and let $\bar{\theta}$ denote the highest-degree homogeneous summand of θ. By what was shown above, $\bar{\theta}y \neq 0$, which implies $\theta y \neq 0$. □

Lemma 6.2. *If $D \in \text{LND}_w(B)$ is non-zero and homogeneous relative to $\mathfrak{g}_2$, then $\ker D \subset \mathbf{k}[w,x,z,t]$.*

Proof. It suffices to assume that D is irreducible. In addition, since $\ker D$ is generated as a $\mathbf{k}$-algebra by homogeneous elements, it suffices to show that $f \in \mathbf{k}[w,x,z,t]$ for each irreducible homogeneous $f \in \ker D$.

Suppose to the contrary that $Df = 0$ for irreducible homogeneous f not in $\mathbf{k}[w,x,z,t]$. We may decompose B as

$$B = \mathbf{k}[w,x,z,t] + y \cdot \mathbf{k}[w,y,z,t] + xy \cdot \mathbf{k}[w,y,z,t] \ .$$

It follows that $\deg_2 f \geq 1$. Let $p \in \mathbf{k}[w,x,z,t]$ and $q, r \in \mathbf{k}[w,y,z,t]$ be such that

$$f = p + yq + xyr \ .$$

Then $\deg_2 p \geq 1$ implies that $p \in wB$. By irreducibility, $\pi_0(D) \neq 0$ and $\pi_0(f) \neq 0$, but $\pi_0(D)(\pi_0(f)) = 0$. Since $\pi_0(f) = \pi_0(yq + xyr) \in yB_0$, it follows that $\pi_0(D)(y) = 0$. Since $\pi_0(D) \in \text{LND}(B_0)$, it follows by *Lemma 6.1* that $\pi_0(D) = 0$, a contradiction.

Therefore, $f \in \mathbf{k}[x,z,t,w]$. □

Proof of Theorem 6.1. Let $D \in \text{LND}(B_1)$ be non-zero, and consider the $\mathbb{Z}$-grading $\mathfrak{g}_2$ of B. We have $B = (B_1)^H$ and $D^H w = 0$. By *Lemma 6.2*, it follows that $\ker(D^H) \subset \mathbf{k}[w,x,z,t]$. Therefore, by *Lemma 3.1*, $\ker D = \pi_1(\ker D^H) \subset \mathbf{k}[x,z,t]$, which implies $\mathcal{D}(B_1) = \mathbf{k}[x,z,t]$. □

Corollary 6.1. $ML(X) = \mathbf{k}[x]$

Proof. Assume that $Dx \neq 0$. Let $f, g \in \ker D \subset \mathbf{k}[x,z,t]$ be algebraically independent, and write

$$f = xf_1 + f_2 \quad \text{and} \quad g = xg_1 + g_2 \ ,$$

where $f_1, g_1 \in \mathbf{k}[x,z,t]$ and $f_2, g_2 \in \mathbf{k}[z,t]$. Then f_2 and g_2 are algebraically independent, since otherwise there is a bivariate polynomial P over $\mathbf{k}$ such that $P(f_2, g_2) = 0$. But then $P(f,g) \in xB$ and is non-zero, meaning $x \in \ker D$, a contradiction.

We have

$$f^H = xwf_1(xw,z,t) + f_2(z,t) \quad \text{and} \quad g^H = xwg_1(xw,z,t) + g_2(z,t) \ ,$$

meaning that $\pi_0(f^H) = f_2$ and $\pi_0(g^H) = g_2$. As these are algebraically independent elements of $\ker \pi_0(D^H)$, it follows that $\pi_0(D^H)(z) = \pi_0(D^H)(t) = 0$. But then $\pi_0(D^H)(x + x^2y) = 0$ implies $\pi_0(D^H) = 0$, contradicting the fact that $D^H(B)$ is not contained in wB. Therefore, $Dx = 0$. □

Corollary 6.2. $|y|_{B_c} = 2$ *for each* $c \in \mathbf{k}$.

Proof. Since $D_1^3 y = 0$, we have $|y|_{B_c} \leq 2$. Let non-zero $D \in \text{LND}(B)$ be given. By *Thm. 2.2*, $D^2(z^2 + t^3) \neq 0$. Since $Dx = 0$, it follows that

$$x^2 D^2 y = D^2(x + x^2 y) = -D^2(z^2 + t^3) \neq 0 \quad \Rightarrow \quad D^2 y \neq 0 \ .$$

Therefore, $|y|_{B_c} = 2$. □

We obtain the following description of the locally nilpotent derivations of $\mathbf{k}[X]$; see also [11], Cor. 9.8.

Corollary 6.3. *$D \in \mathrm{LND}(\mathbf{k}[X])$ and $D \neq 0$ if and only if there exists $F \in \mathbf{k}[x,z,t]$ such that F is a $\mathbf{k}(x)$-variable of $\mathbf{k}(x)[z,t]$, and $D = F_t D_1 - F_z D_2$.*

Note that D_1 and D_2 commute.

Proof. First, by *Lemma 2.2* and *Thm. 6.1*, any $D \in \mathrm{LND}(\mathbf{k}[X])$ restricts to a locally nilpotent derivation of $\mathbf{k}[x,z,t]$, and by *Cor. 6.1*, we have $Dx = 0$. We show that the restriction induces a bijection between the sets

$$\mathrm{LND}(\mathbf{k}[X]) \quad \text{and} \quad x^2 \cdot \mathrm{LND}_{\mathbf{k}[x]}(\mathbf{k}[x,z,t]) \ .$$

Suppose that $D \in \mathrm{LND}(\mathbf{k}[X])$, and let $\overline{D}$ be the induced locally nilpotent derivation of $\mathbf{k}[z,t]$ given by $\overline{D}f \equiv Df \mod (x)$. We have that $D(z^2+t^3)$ is in $x^2\mathbf{k}[X] \cap \mathbf{k}[x,z,t]$. In particular, $\overline{D}(z^2+t^3) = 0$. Thus, by *Cor. 2.1*, $\overline{D} = 0$, and therefore $D = xD_0$, where $D_0 \in \mathrm{LND}(\mathbf{k}[X])$. By the same argument, we find that the image $D_0(\mathbf{k}[z,t])$ is in the ideal (x), and thus the restriction of D to $\mathbf{k}[x,z,t]$ belongs to $x^2 \cdot \mathrm{LND}_{\mathbf{k}[x]}(\mathbf{k}[x,z,t])$. Finally, for any $\Delta \in x^2 \cdot \mathrm{LND}_{\mathbf{k}[x]}(\mathbf{k}[x,z,t])$, we can extend Δ to a locally nilpotent derivation on $\mathbf{k}[X]$ by setting $\Delta y = -\Delta(z^2+t^3)/x^2$.

Note that the restrictions of D_1 and D_2 to $\mathbf{k}[x,z,t]$ are simply $x^2\partial_z$ and $x^2\partial_t$, respectively. Rentschler's theorem ([11], Theorem 4.1) implies that any locally nilpotent derivation of $\mathbf{k}[x,z,t]$ which has x in the kernel is of the form $F_t\partial_z - F_z\partial_t$, where $F \in \mathbf{k}[x,z,t]$ is a $\mathbf{k}(x)$-variable of $\mathbf{k}(x)[z,t]$. This proves the corollary. □

7. $\mathbb{Z}$-Gradings of Polynomial Rings

For rings of polynomials which are $\mathbb{Z}$-graded over a ground ring, the main theorem (*Thm. 4.1*) provides a surprisingly simple and effective degree criterion which implies that certain variables are either local slices or invariants for *all* homogeneous locally nilpotent derivations. *Theorem 7.1* below shows that, from a finite sequence of integers, we can immediately deduce invariant properties of homogeneous derivations which are otherwise difficult to calculate. Further, it is important to note that the theorem does not assume that the derivation D is an A-derivation, only that the grading is an A-grading – the result applies to *all* homogeneous locally nilpotent derivations of the polynomial ring.

Theorem 7.1. *Suppose A is a commutative $\mathbf{k}$-domain, and $B = A[x_1, ..., x_n] = A^{[n]}$ is $\mathbb{Z}$-graded over A, where each x_i is homogeneous of degree a_i ($1 \leq i \leq n$), and $\gcd(a_1, ..., a_n) = 1$. Let homogeneous $D \in \mathrm{LND}(B)$ be given.*

- **(a)** *If $\gcd(a_1, ..., \hat{a_i}, ..., a_n) \neq 1$ for some i, then $D^2 x_i = 0$.*
- **(b)** *If $\gcd(a_1, ..., \hat{a_i}, ..., a_n) \neq 1$ and $\gcd(a_1, ..., \hat{a_j}, ..., a_n) \neq 1$ for $i \neq j$, then $Dx_i = 0$ or $Dx_j = 0$.*

Proof. Let homogeneous $D \in \mathrm{LND}(B)$ be given. For convenience, assume $\gcd(a_1, ..., a_{n-1}) \neq 1$, and let $p \in \mathbb{Z}$ be a prime dividing each a_j for $j = 1, ..., n-1$. Set $R = A[x_1, ..., x_{n-1}, y] = A^{[n]}$, where $\deg x_j = a_j$ for $1 \leq j \leq n-1$, and $\deg y = pa_n$. Then $B = R[x_n]/(y + x_n^p)$ has a natural $\mathbb{Z}_p$-grading over R such that $\deg x_n = a_n$. Since this is the same $\mathbb{Z}_p$-grading induced by the given $\mathbb{Z}$-grading, it follows that D is homogeneous relative to this $\mathbb{Z}_p$-grading of B. By *Thm. 4.1*, $D^2 x_n = 0$. This proves part (a).

Under the additional hypotheses of (b), and still assuming $i = n$, suppose that $Dx_n \neq 0$. Then by *Thm. 4.1*, we have that $D|_R = x_n^{p-1}\delta$ for some $\delta \in \mathrm{LND}(R)$. It follows that $Dx_j = x_n^{p-1}\delta x_j \in \ker D$. Since $x_n \notin \ker D$, it follows that $\delta x_j = 0$. Therefore, $Dx_j = 0$ when $Dx_n \neq 0$. By symmetry, $Dx_n = 0$ when $Dx_j \neq 0$. This proves part (b). □

In the particular case of two variables adjoined to A, *Thm. 7.1* gives the following result.

Corollary 7.1. *Suppose A is a commutative $\mathbf{k}$-domain, and $B = A[x,y] = A^{[2]}$. Assume B is $\mathbb{Z}$-graded over A, where x, y are homogeneous, $\deg x = a$, $\deg y = b$, and $\gcd(a,b) = 1$. If $|a|, |b| \geq 2$, then for every homogeneous $D \in \mathrm{LND}(B)$, either $Dx = 0$ or $Dy = 0$.*

Proof. This is immediately implied by part (b) of *Thm. 7.1.* □

Part of the recent thesis of Kolhatkar [14] investigates locally nilpotent derivations of polynomial rings $\mathbf{k}[x_1, ..., x_n]$ which are homogeneous relative to gradings by an abelian group G, with special interest in the case $G = \mathbb{Z}$. (Her results assume that $\mathbf{k}$ is algebraically closed.) In particular, suppose a linear $\mathbb{Z}$-grading $\mathfrak{g}$ of B is defined by $\deg x_i = a_i \in \mathbb{Z}$ for $i = 1, ..., n$, where $\gcd(a_1, ..., a_n) = 1$. Given i, set $\alpha_i = \gcd(a_1, ..., \hat{a_i}, ..., a_n)$. Then the *type* of $\mathfrak{g}$ is defined by

$$\text{type}(\mathfrak{g}) = \#\{i \mid \alpha_i \neq 1\} \ .$$

In Section 2.3.14 of her thesis, Kolhatkar shows that, if $D \in \text{LND}(B)$ is homogeneous, then

$$\text{rank}(D) + \text{type}(\mathfrak{g}) \leq n + 1 \ .$$

The integer $\text{rank}(D)$ is defined in Sect. 3.2.1 of [11], and coincides with $\text{rank}_{\mathbf{k}}(D)$ as defined below.

In order to extend Kolhatkar's result, let A be a commutative $\mathbf{k}$-domain and $B = A[x_1, ..., x_n] = A^{[n]}$ for $n \geq 2$. Let $\mathfrak{g}$ be a $\mathbb{Z}$-grading of B over A, where each x_i is homogeneous, $\deg x_i = a_i$ for $i = 1, ..., n$, and $\gcd(a_1, ..., a_n) = 1$. Given i, set $\alpha_i = \gcd(a_1, ..., \hat{a_i}, ..., a_n)$. Then the *type* over A of $\mathfrak{g}$ is defined by

$$\text{type}_A(\mathfrak{g}) = \#\{i \mid \alpha_i \neq 1\} \ .$$

Similarly, given $D \in \text{LND}_A(B)$, the *corank* of D over A is the largest integer m such that there exist $y_1, ..., y_m \in B$ satisfying

$$y_1, ..., y_m \in \ker D \quad \text{and} \quad B = A[y_1, ..., y_m]^{[n-m]} \ ,$$

denoted $\text{corank}_A(D)$; and the *rank* of D over A is defined by $\text{rank}_A(D) = n - \text{corank}_A(D)$. Part (b) of *Thm. 7.1* implies:

Corollary 7.2. *Suppose A is a commutative $\mathbf{k}$-domain, $B = A^{[n]}$, and $\mathfrak{g}$ is a linear $\mathbb{Z}$-grading of B over A. Given homogeneous $D \in \text{LND}_A(B)$,*

$$\text{rank}_A(D) + \text{type}_A(\mathfrak{g}) \leq n + 1 \ .$$

The following example illustrates that Kolhatkar's bound cannot, in general, be improved.

Example 7.1. For even $n \geq 6$, consider the polynomial ring $R = \mathbf{k}[x_1, ..., x_n] = \mathbf{k}^{[n]}$ with the standard $\mathbb{Z}$-grading defined by $\deg x_i = 1$ for each i. By Thm. 3.37 of [11], there exists homogeneous $\delta \in \text{LND}(R)$ of rank n and degree 4 such that $\delta^2 x_i = 0$ for each i.

For prime $p \in \mathbb{Z}$, define $B = R[z]/(x_n + z^p) = \mathbf{k}[x_1, ..., x_{n-1}, z]$. Then B has a $\mathbb{Z}$-grading $\mathfrak{g}$ defined by $\deg x_i = p$ $(1 \leq i \leq n)$ and $\deg z = 1$. In particular, $\text{type}(\mathfrak{g}) = 1$. Define homogeneous $D \in \text{LND}(B)$ as in the proof of *Lemma 4.1* above:

$$Dr = z^{p-1}\delta r \quad (r \in R) \quad \text{and} \quad Dz = -\tfrac{1}{p}\delta x_n \ .$$

Note that $\delta x_n \neq 0$, since the rank of δ is n. Therefore, $Dz \neq 0$. It remains to show that the rank of D is n.

Assume $v \in B$ is a variable such that $Dv = 0$, and let L be its linear part, i.e., L is the degree-one summand of v relative to the standard $\mathbb{Z}$-grading of B. Then $L = c_1x_1 + \cdots c_{n-1}x_{n-1} + c_nz$ for $c_i \in \mathbf{k}$, and since v is a variable, $L \neq 0$.

Next, consider the $\mathbb{Z}_p$-grading of B induced by $\mathfrak{g}$. Since D is $\mathbb{Z}_p$-homogeneous and $Dz \neq 0$, *Thm. 4.1(b)* implies that $\ker D = \ker \delta$. In particular, $v \in \ker \delta$. It follows that $c_n = 0$, since v can only support p-th powers of z. Since δ is homogeneous in the standard grading of R, it follows that $L \in \ker \delta$, contradicting the fact that δ has no variable in its kernel.

Therefore, the rank of D is n.

8. Pham-Brieskorn Varieties

Given $n \geq 1$ and integers $a_i \geq 2$, $0 \leq i \leq n$, the corresponding *Pham-Brieskorn variety* is the hypersurface $H \subset \mathbb{A}^{n+1}$ defined by

$$x_0^{a_0} + x_1^{a_1} + \cdots + x_n^{a_n} = 0 \ .$$

These hypersurfaces have been of interest in topology and algebraic geometry for decades; see for example the excellent survey of Seade [20].

8.1. Pham-Brieskorn Surfaces.

Given positive integers a, b, c, the corresponding Pham-Brieskorn surface is defined by

$$x^a + y^b + z^c = 0$$

in $\mathbb{A}^3$, and is denoted $S_{(a,b,c)}$.

Define $T \subset \mathbb{Z}^3$ to be the set of triples (a, b, c) satisfying $a, b, c \geq 2$, and at most one of a, b, c equals 2. Then $T = T_1 \cup T_2$ for subsets T_1 and T_2 defined by:

$$T_1 = \{(a, b, c) \in T \mid \gcd(ab, c) = 1 \text{ or } \gcd(ac, b) = 1 \text{ or } \gcd(bc, a) = 1\}$$
$$T_2 = \{(a, b, c) \in T \mid a^{-1} + b^{-1} + c^{-1} \leq 1\}.$$

The following is a version of Mason's Theorem. We say that $f, g \in B$ are *relatively prime* if and only if $fB \cap gB = fgB$.

Theorem 8.1. *Let B be a commutative $\mathbf{k}$-domain, and suppose $x, y, z \in B$ are pairwise relatively prime elements which satisfy $x^a + y^b + z^c = 0$ for integers $a, b, c \geq 2$.*

- **(a)** *If $(a, b, c) \in T_2$, then $\mathbf{k}[x, y, z] \subset ML(B)$.*
- **(b)** *If $D \in \text{LND}(B)$ and $\nu_D(z) = 1$, then $a = b = 2$.*

Proof. Assume that $\mathbf{k}[x, y, z]$ is not contained in $ML(B)$, and let $D \in \text{LND}(B)$ be given, where at least one of Dx, Dy, Dz is non-zero.

Clearly, if two of Dx, Dy, Dz are 0, then the third is also zero. So consider the case $Dz = 0$, but $Dx \neq 0$ and $Dy \neq 0$. Then $D(x^a + y^b) = 0$. By *Thm. 2.2 (a)*, it follows that $Dx = Dy = 0$, which contradicts the assumption. Therefore, $\nu_D(x)$, $\nu_D(y)$, and $\nu_D(z)$ are positive.

Apply D to the equation $x^a + y^b + z^c = 0$ to obtain

$$\begin{pmatrix} x & y & z \\ aDx & bDy & cDz \end{pmatrix} \begin{pmatrix} x^{a-1} \\ y^{b-1} \\ z^{c-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ .$$

Define the matrix

$$A = \begin{pmatrix} x & y & z \\ aDx & bDy & cDz \\ 0 & 0 & 1 \end{pmatrix} \ .$$

If $\det(A) = bxDy - ayDx = 0$, then either $Dx = 0$ or $Dy = 0$ ([11], Princ. 5), a contradiction. Therefore $\det(A) \neq 0$. We have

$$A \begin{pmatrix} x^{a-1} \\ y^{b-1} \\ z^{c-1} \end{pmatrix} = z^{c-1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \det(A) \begin{pmatrix} x^{a-1} \\ y^{b-1} \\ z^{c-1} \end{pmatrix} = z^{c-1} \text{adj}(A) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = z^{c-1} \begin{pmatrix} cyDz - bzDy \\ azDx - cxDz \\ bxDy - ayDx \end{pmatrix} \ .$$

Since x, y, and z are pairwise relatively prime in B, it follows that:

$$z^{c-1} \text{ divides } \det(A) = bxDy - ayDx \quad \Rightarrow \quad (c-1)\nu_D(z) \leq \nu_D(x) + \nu_D(y) - 1$$
$$y^{b-1} \text{ divides } azDx - cxDz \quad \Rightarrow \quad (b-1)\nu_D(y) \leq \nu_D(x) + \nu_D(z) - 1$$
$$x^{a-1} \text{ divides } cyDz - bzDy \quad \Rightarrow \quad (a-1)\nu_D(x) \leq \nu_D(y) + \nu_D(z) - 1$$

In order to prove part (a), let $\sigma = \nu_D(x) + \nu_D(y) + \nu_D(z)$. The inequalities above show that

$$\nu_D(x) \leq \frac{\sigma - 1}{a} \ , \quad \nu_D(y) \leq \frac{\sigma - 1}{b} \ , \quad \nu_D(z) \leq \frac{\sigma - 1}{c} \ .$$

By addition, we obtain

$$\sigma \leq \frac{\sigma - 1}{a} + \frac{\sigma - 1}{b} + \frac{\sigma - 1}{c} = (\sigma - 1)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \ .$$

This implies

$$1 < 1 + \frac{1}{\sigma - 1} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \ ,$$

i.e., $(a, b, c) \notin T_2$. This proves part (a).

For part (b), assume that $\nu_D(z) = 1$. Then the inequalities above yield

$$1 = \nu_D(z) \geq (b-1)\nu_D(y) - \nu_D(x) + 1 \quad \text{and} \quad 1 = \nu_D(z) \geq (a-1)\nu_D(x) - \nu_D(y) + 1 \ ,$$

which in turn give

$$\nu_D(x) \geq (b-1)\nu_D(y) \quad \text{and} \quad \nu_D(y) \geq (a-1)\nu_D(x) \ .$$

Therefore, $\nu_D(y) \geq (a-1)(b-1)\nu_D(y)$, which implies $a = b = 2$, since $\nu_D(y) \neq 0$. This proves part (b). □

As an immediate corollary, we have:

Corollary 8.1. *Let A be a commutative $\mathbf{k}$-domain, and suppose that $A[x, y, z] = A^{[3]}$. If $(a, b, c) \in T_2$ and $B = A[x, y, z]/(x^a + y^b + z^c)$, then $\text{LND}_A(B) = \{0\}$.*

Theorem 8.2. *Suppose positive integers a, b, c are given.*

- **(a)** *If $(a, b, c) \in T$, then $S_{(a,b,c)}$ is rigid.*
- **(b)** *If $(a, b, c) \in T_2$, then $S_{(a,b,c)}$ is stably rigid.*

Proof. Let $B = \mathbf{k}[S_{(a,b,c)}] = \mathbf{k}[x, y, z]/(x^a + y^b + z^c)$. For any integer $n \geq 0$, suppose $D \in \text{LND}(B^{[n]})$ is given. If $(a, b, c) \in T_2$, then $Dx = Dy = Dz = 0$ by *Thm. 8.1*, meaning $DB = 0$. This proves part (b).

Therefore, in order to prove part (a), it suffices to assume that $(a, b, c) \in T_1$ and $n = 0$. In particular, suppose $\gcd(ab, c) = 1$, and define a $\mathbb{Z}$-grading of B by

$$\deg x = bc \ , \quad \deg y = ac \ , \quad \deg z = ab \ .$$

Then $R = \mathbf{k}[x, y]$ is a homogeneous subring, and $f = x^a + y^b$ is a homogeneous element of R of degree ab relatively prime to c. Since $|f|_R \geq 2$ by *Cor. 2.1*, it follows by *Cor. 4.2* that B is rigid. This proves part (a). □

Corollary 8.2. *Given polynomials $f, g, h \in \mathbf{k}[t] = \mathbf{k}^{[1]}$, let*

$$B = \mathbf{k}[x, y, z]/(f(x) + g(y) + h(z)) \ .$$

- **(a)** *If $(\deg f, \deg g, \deg h) \in T$, then B is rigid.*
- **(b)** *If $(\deg f, \deg g, \deg h) \in T_2$, then B is stably rigid.*

Proof. With no loss of generality, we may assume f, g, h are monic. Set

$$a = \deg f \ , \quad b = \deg g \ , \quad \text{and} \quad \deg h = c \ .$$

Let $Q = \mathbf{k}[x, y, z] = \mathbf{k}^{[3]}$, and define a linear $\mathbb{Z}$-grading $\mathfrak{g}$ of Q by setting $\deg x = bc$,$\deg y = ac$, and $\deg z = ab$. Let I be the principal ideal of Q defined by $I = (f(x) + g(y) + h(z))$. Then the triple $(Q, \mathfrak{g}, I)$ defines the $\mathbb{Z}$-graded ring B^H, and to each $D \in \text{LND}(B)$ we associate $D^H \in \text{LND}(B^H)$ such that D^H modulo $(w - 1)$ equals D. Since $D^H w = 0$, the quotient derivation d defined by D^H modulo (w) on B^H/wB^H is locally nilpotent. If $D \neq 0$, then D^H is irreducible, and therefore $d \neq 0$. However

$$B^H/wB^H = \mathbf{k}[x, y, z]/(x^a + y^b + z^c)$$

is rigid (respectively, stably rigid) when $(a,b,c) \in T$ (respectively, $(a,b,c) \in T_2$), contradicting the fact that $d \neq 0$. Therefore, $D = 0$. □

Question. Are the surfaces $S_{(2,3,3)}$, $S_{(2,3,4)}$, and $S_{(2,3,5)}$ stably rigid?

Remark 8.1. If R is a rigid affine $\mathbf{k}$-domain, then it is known that $ML(R^{[1]}) = R$. Moreover, if $\dim_{\mathbf{k}} R = 1$, then $ML(R^{[n]}) = R$ for all $n \geq 0$, i.e., R is stably rigid. In general, it remains an open question whether $ML(R^{[2]}) = R$ when $\dim_{\mathbf{k}} R \geq 2$; some cases were settled by Crachiola in his thesis. See [2] and [11] for details.

Remark 8.2. *Thm. 8.1* is false for $(a,b,c) \in T_1 - T_2$. The set $T_1 - T_2$ consists of the three elements $(2,3,3), (2,3,4)$, and $(2,3,5)$, and for each of these there exist pairwise relatively prime $x(t), y(t), z(t) \in \mathbf{k}[t]$ such that $x(t)^a + y(t)^b + z(t)^c = 0$ (see [10]). There are even counterexamples if we add the condition that x, y, z should be pairwise algebraically independent. For example, [6] gives $x(u,v), y(u,v), z(u,v) \in \mathbf{k}[u,v] = \mathbf{k}^{[2]}$ which are pairwise algebraically independent such that $x(u,v)^2 + y(u,v)^3 + z(u,v)^5 = 0$. In addition, *Thm. 8.1* is false without the assumption x, y, z are pairwise relatively prime. For example, if $\zeta \in \mathbb{C}$ is a primitive ninth root of unity and $\mathbb{C}[t] = \mathbb{C}^{[1]}$, then $t^3 + (\zeta t)^3 + (\zeta^2 t)^3 = 0$.

Remark 8.3. In [9], Lemma 2, the authors give a result identical to part (a) of *Thm. 8.1*, but without the assumption that x, y, z are pairwise relatively prime. Thus, their result is not correct as stated. However, in their applications, the elements involved are pairwise relatively prime, so their subsequent results remain valid.

Remark 8.4. *Theorem 8.2* was essentially proved by Kaliman and Zaidenberg in [13]. They do not explicitly state the stable rigidity of $S = S_{(a,b,c)}$ when $(a,b,c) \in T_2$, but this is contained in their proof, which relies on geometric methods in addition to Mason's Theorem. They consider the existence of sufficiently many rational curves on a variety, and show in particular that any rational curve on S must pass through the singularity. If there were a non-trivial $\mathbb{G}_a$-action on $V = S \times \mathbb{A}^n$, then any one-dimensional orbit not contained in $\mathrm{Sing}(V)$ must be disjoint from $\mathrm{Sing}(V)$. The projection of this orbit to S is thus either a point or a rational curve not passing through the singularity. So the only possibility is that this projection is a point.

8.2. **Pham-Brieskorn Threefolds.**

Theorem 8.3. *Let a, b, c, d be integers such that $(a,b,c) \in T$, $d \geq 2$, and $\gcd(abc, d) = 1$. Then the ring*

$$B = \mathbf{k}[x,y,z,t]/(x^a + y^b + z^c + t^d)$$

is rigid.

Proof. Define a $\mathbb{Z}$-grading of B by

$$\deg x = bcd\ , \quad \deg y = acd\ , \quad \deg z = abd\ , \quad \deg t = abc\ .$$

Then $R = \mathbf{k}[x,y,z]$ is a homogeneous subring, and $f = x^a + y^b + z^c$ is a homogeneous element of R of degree abc relatively prime to d. Since $|f|_R \geq 2$ by *Thm. 2.4*, it follows by *Cor. 4.2* that B is rigid. □

Remark 8.5. An important open question is whether the Pham-Brieskorn cubic threefold defined by

$$x^3 + y^3 + z^3 + t^3 = 0$$

is rigid.

We also point out the following cubic threefold.

Theorem 8.4. *If B is the ring*

$$B = \mathbf{k}[x, y, z, t]/(x^3 + y^3 + xyz + t^3) \ ,$$

then $ML(B) = \mathbf{k}$.

Proof. Let $R = \mathbf{k}[x, y, z] = \mathbf{k}^{[3]}$, and define $r \in R$ by $r = x^3 + y^3 + xyz$. According to Section 5.5.2 of [11], there exists a sequence $\delta_n \in \mathrm{LND}(R)$ such that $\delta_n r \in \ker \delta_n$, but $\delta_n r \neq 0$, and such that $\ker \delta_m \cap \ker \delta_n = \mathbf{k}$ when $n \geq m + 2$. By *Cor. 4.1*, it follows that there exists a sequence $D_n \in \mathrm{LND}(B)$ such that $\ker D_n = \ker \delta_n$ for each n. We conclude that $ML(B) = \mathbf{k}$. □

Remark 8.6. The same argument shows that every fiber $x^3 + y^3 + xyz + t^3 = \lambda$ for $\lambda \in \mathbf{k}$ has trivial Makar-Limanov invariant.

9. Pham-Brieskorn Surfaces with Parameter

In this section, we consider threefolds defined by rings of the form

$$B = R[x, y, z]/(ux^a + vy^b + wz^c) \ ,$$

where $R = \mathbf{k}[t] = \mathbf{k}^{[1]}$ and $u, v, w \in R$ satisfy $\gcd(u, v, w) = 1$. These may be viewed as Pham-Brieskorn surfaces with a parameter introduced, in the sense that $\mathbf{k}(t) \otimes_{\mathbf{k}} B$ is the coordinate ring of a Pham-Brieskorn surface over the function field $\mathbf{k}(t)$.

We first need the following result, which generalizes Thm. 5.1 of [3].

Lemma 9.1. *Given integers a, b, c with $a, b, c \geq 2$, the ring*

$$B = \mathbf{k}[x, y, z]/(x^a y^b + z^c) \ .$$

is rigid.

Proof. We may assume that the ground field $\mathbf{k}$ is algebraically closed. Note that B is a reduced ring, and the surface $X = \mathrm{Spec}(B)$ is irreducible if and only if $\gcd(a, b, c) = 1$ (see remark following proof).

Consider first the case $\gcd(a, b, c) = 1$. Let $D \in \mathrm{LND}(B)$ be given . If $D^2(x^a y^b) = 0$, then $a, b \geq 2$ implies that xy divides the image $D(x^a y^b)$, which is in the kernel of D. But then $Dx = Dy = 0$, which implies $D = 0$. Therefore, $|x^a y^b|_B \geq 2$. Since $\gcd(a, b, c) = 1$, there exists a $\mathbb{Z}$-grading of B such that x, y, z are homogeneous and c is relatively prime to $\deg(x^a y^b)$. By *Cor. 4.2*, we conclude that B is rigid.

For the general case, set $e = \gcd(a, b, c)$, and assume $e \geq 2$. Then F is a reducible polynomial in which each prime factor appears with multiplicity one. Any $\mathbb{G}_a$-action on $X = \mathrm{Spec}(B)$ restricts to each of the irreducible components $X_1, ..., X_e$ of X (see [3], Prop. 1.4). In addition, any $\mathbb{G}_a$-action on X must fix the intersection $Y = \cap_{1 \leq i \leq e} X_i$, which is a union of two distinct but intersecting lines (since $e \geq 2$). Therefore, any $\mathbb{G}_a$-action on X restricts to the complement $X_i - Y$ for each i.

The component X_i has the form $X_i = \mathrm{Spec}(B_i)$, where

$$B_i = \mathbf{k}[x, y, z]/(x^{a/e} y^{b/e} + \lambda_i z^{c/e}) \quad (\lambda_i \in \mathbf{k}^*) \ .$$

If $a/e, b/e, c/e \geq 2$, then by what was shown above, each B_i is rigid, so any $\mathbb{G}_a$-action on X is trivial. Otherwise, $a/e = 1$, $b/e = 1$, or $c/e = 1$, meaning that each component X_i is isomorphic to a Danielewski surface (possibly a plane). Let i be given, $1 \leq i \leq e$. Since $Y \subset X_i$ is defined by $xy = z = 0$, we see that $X_i - Y$ is isomorphic to $\mathbf{k}^* \times \mathbf{k}^*$, which is a rigid variety. Therefore, the only $\mathbb{G}_a$-action on X is trivial. □

Remark 9.1. The ring B in the preceding lemma is a domain if and only if $\gcd(a, b, c) = 1$. To see this, define $F \in \mathbf{k}[x, y, z] = \mathbf{k}^{[3]}$ by $F = x^a y^b + z^c$. Set $d = \gcd(a, b)$, $\alpha = a/d$, and $\beta = b/d$. Then $F = (x^\alpha y^\beta)^d + z^c$. If $\gcd(a, b, c) \geq 2$, then F is reducible and B is not a domain. If $\gcd(a, b, c) = 1$, then $\gcd(d, c) = 1$ and the polynomial $X^d + Y^c$ is irreducible in $\mathbf{k}[X, Y] = \mathbf{k}^{[2]}$, meaning that F is reducible if and only if $x^\alpha y^\beta$ and z^c have a common factor in $\mathbf{k}[x, y, z]$. Therefore, F is irreducible, and B is a domain in this case.

9.1. **The Case of One Coefficient with Parameter.**

Theorem 9.1. *Given integers $a, b, c, d \geq 2$, define*

$$B = \mathbf{k}[t, x, y, z]/(t^d x^a + y^b + z^c) \ .$$

Set $e = \gcd(a, d)$. Then B is rigid if either (1) $(e, b, c) \in T$, or (2) $e = 2$, $a \neq 2$, $d \neq 2$, and b, c are not both 2.

Proof. Set $R = \mathbf{k}[y, z]$ and $f = y^b + z^c$. Let m, n be positive integers such that $a = em$ and $d = en$. We have

$$B = R[t, x]/(f + t^d x^a) = R[t, x, v]/(f + v^e, v - t^n x^m) = S[t, x]/(v - t^n x^m) \ ,$$

where

$$S = R[v]/(f + v^e) = \mathbf{k}[v, y, z]/(v^e + y^b + z^c) \ .$$

If $|v|_S = 1$, then *Thm. 8.1(b)* implies that $b = c = 2$, a contradiction. If $|v|_S = 0$, then the quotient ring R/fR is not rigid, which is again a contradiction. Therefore, $|v|_S \geq 2$.

If $e \neq 2$, or if $e = 2$ and $b, c \geq 3$, then S is rigid by *Thm. 8.2*. It follows from *Cor. 4.6* that B is rigid in this case.

Otherwise, $e = 2$ and $m, n \geq 2$. Since $|v|_S \geq 2$, it follows from *Cor. 4.5* that B is rigid in this case as well. □

Remark 9.2. When $a, b, c, d \geq 2$, all remaining cases are non-rigid. They are of the form

$$t^d x^a + y^2 + z^2 \quad \text{or} \quad t^{2n} x^2 + y^2 + z^c \ ,$$

and can be managed as in the following example. It should be noted that when $d = 1$, the hypersurface $tx^a + y^b + z^c = 0$ is non-rigid for all positive integers a, b, c.

Theorem 9.2. *Given an odd integer $n \geq 1$, define the ring*

$$B = \mathbf{k}[t, x, y, z]/(t^2 x^2 + y^2 + z^n)$$

If $i = \sqrt{-1}$ belongs to $\mathbf{k}$, then $ML(B) = \mathbf{k}$.

Proof. If $n = 1$, then $B = \mathbf{k}^{[3]}$, and the result is clear. So assume $n \geq 3$.

Let $R = \mathbf{k}[t, x, y]$, and define $f \in R$ by $f = t^{2k} x^2 + y^2$. If $u = y + it^k x$ and $v = y - it^k x$, then $R = \mathbf{k}[t, x, u] = \mathbf{k}[t, x, v]$ and $f = uv$. Define $\delta_1, \delta_2 \in \text{LND}(R)$ by

$$\delta_1(t) = 1 \, , \ \delta_1(x) = \delta_1(u) = 0 \quad \text{and} \quad \delta_2(x) = 1 \, , \ \delta_2(t) = \delta_2(u) = 0 \ .$$

Then $\delta_1^2 f = \delta_2^2 f = 0$. By the construction in *Lemma 4.1*, there exist $D_1, D_2 \in \text{LND}(B)$ such that D_j is a quasi-extension of δ_j $(j = 1, 2)$. In addition,

$$\ker D_1 = \ker \delta_1 = \mathbf{k}[x, u] \quad \text{and} \quad \ker D_2 = \ker \delta_2 = \mathbf{k}[t, u] \ ,$$

so $ML(B) \subset \mathbf{k}[u]$. By symmetry, $ML(B) \subset \mathbf{k}[v]$ as well. Therefore, $ML(B) = \mathbf{k}$. □

9.2. **The Case of Two Coefficients with Parameter.**

Theorem 9.3. *Suppose $a, b, c, d, e \geq 2$, and set*

$$B = \mathbf{k}[t, x, y, z]/(t^d x^a + t^e y^b + z^c) \ .$$

- **(a)** *$t \in ML(B)$.*
- **(b)** *If $(a, b, c) \in T$, then B is rigid.*

Proof. Since $d, e \geq 2$, the singular locus of $X = \text{Spec}(B)$ consists of the union of the hypersurface $Y \subset X$ defined by the ideal tB and the line defined by $x = y = 0$. Therefore, any $\mathbb{G}_a$-action on X restricts to Y, meaning that tB is an integral ideal for each $D \in \text{LND}(B)$. Consequently, $Dt = 0$ for every $D \in \text{LND}(B)$. This proves part (a).

For part (b), assume $(a, b, c) \in T$. If $\text{LND}(B) \neq \{0\}$, then $|t|_B = 0$, which implies that the quotient $B/(t-1)B$ is non-rigid, contradicting *Thm. 8.2*. Therefore, B is rigid. □.

Remark 9.3. Without the hypothesis that $d, e \geq 2$ in this theorem, we do not know whether B is rigid when $(a, b, c) \in T$. Some of these cases can be settled using the following lemma combined with *Cor. 4.2*. It should be noted that when $d = e = 1$, the hypersurface $tx^a + ty^b + z^c = 0$ is non-rigid for all positive integers a, b, c.

Lemma 9.2. *Let $R = \mathbf{k}[t, x, y] = \mathbf{k}^{[3]}$, and let $f \in R$ be given by*

$$f = t^d x^a + t^e y^b \ ,$$

where $d, e \geq 1$, $a, b \geq 2$ and a, b are not both 2. Then $|f|_R \geq 2$.

Proof. We may assume, with no loss of generality, that $d \geq e$. Suppose that $|f|_R \leq 1$, and choose irreducible $D \in \mathrm{LND}(R)$ with $|f|_R = \nu_D(f)$. If $Dt = 0$, then $\nu_D(f) \geq 2$ by *Thm. 2.1*, a contradiction. Therefore, $Dt \neq 0$, which means $\nu_D(t) \geq 1$. We have

$$1 \geq \nu_D(f) = e\nu_D(t) + \nu_D(t^{d-e}x^a + y^b) \ ,$$

which implies

$$e = \nu_D(t) = 1 \quad \text{and} \quad \nu_D(t^{d-e}x^a + y^b) = 0 \ .$$

It follows that $f = tg$ for $g = t^{d-1}x^a + y^b$, $D^2 t = 0$, and $Dg = 0$. If $d = 1$, then by *Thm. 2.1* it follows that $Dx = Dy = 0$. But then $D = 0$, a contradiction. Therefore, $d \geq 2$.

Since D is irreducible, it induces a non-zero locally nilpotent derivation θ on the quotient ring $\bar{R} = \mathbf{k}[t, x, y]/(g)$. In particular, $\bar{R}$ is not rigid. If $d \geq 3$, then *Lemma 9.1* implies that $\bar{R}$ is rigid, a contradiction. Therefore, $d = 2$.

We thus have $f = tg$ for $g = tx^a + y^b$, where $D^2 t = 0$ and $Dg = 0$. This implies $\theta^2 t = 0$. Since $ML(\bar{R}) = \mathbf{k}[x]$, it follows that $\theta x = 0$ (see [15, 16]), and $\theta y \neq 0$ (otherwise $\theta = 0$). But then

$$1 \geq \nu_\theta(t) = \nu_\theta(tx^a) = \nu_\theta(y^b) = b\nu_\theta(y) \geq b \geq 2 \ ,$$

a contradiction. Therefore $|f|_R \geq 2$. □

In their paper [12], Kaliman and Makar-Limanov consider the complex threefolds defined by

$$t^{m(n-1)}x^n - t^{m(k-1)}y^k + z^\ell = 0 \ ,$$

where $m \geq 1$, $n > k \geq 2$, $\gcd(n, k) = 1$, and $\ell \geq 2$. They show that such a threefold is rigid, except when $k = l = 2$ and m is even. Our framework provides an alternate proof over any field $\mathbf{k}$ of characteristic zero.

Corollary 9.1. (see [12], Prop. 10.1) *Let*

$$B = \mathbf{k}[t, x, y, z]/(t^{m(n-1)}x^n - t^{m(k-1)}y^k + z^\ell) \ ,$$

where $\ell \geq 2$, $m \geq 1$, $n \geq k \geq 2$, and n, k are not both 2. Then B is rigid except in the case $k = \ell = 2$ and m is even.

Proof. Set $\gamma = \gcd(\ell, m)$. Consider first the case $\gamma = 1$. In this case, define a $\mathbb{Z}$-grading on $R = \mathbf{k}[t, x, y] = \mathbf{k}^{[3]}$ by

$$(\deg t, \deg x, \deg y) = (-1, m, m) \ .$$

Then $f \in R$ is homogeneous of degree m, and ℓ is relatively prime to m. *Lemma 9.2* and *Cor. 4.2* imply that B is rigid.

So assume that $\gamma \geq 2$. Then

$$m \geq \gamma \geq 2 \quad \Rightarrow \quad m(n-1) \geq m(k-1) \geq 2 \ .$$

By *Thm. 9.3*, if $(\ell, n, k) \in T$, then B is rigid. Otherwise, the triple (ℓ, n, k) equals $(2, 2, N)$, $(2, N, 2)$, or $(N, 2, 2)$ for some $N \geq 2$. As $k = \min\{k, n\}$, we conclude that $k = 2$, and therefore $n \neq 2$. Thus, $\ell = \gamma = 2$ and 2 divides m. □

10. Concluding Remarks

Remark 10.1. Let B be an affine $\mathbf{k}$-domain with $\dim_{\mathbf{k}} B \geq 2$. Then for each $f \in B$ with $|f|_B = 0$, the ring B/fB is non-rigid. This leads naturally to the question:

If $f \in B$ and $|f|_B \geq 1$, can B/fB be non-rigid?

So far, we know of no examples where this occurs. The main case of interest is when B is a polynomial ring over $\mathbf{k}$. The following lemma pertains to a special case in which this question can be settled.

Lemma 10.1. *Let $Q = \mathbf{k}^{[n]}$ for $n \geq 1$, and let $f \in Q$ be such that $|f|_Q \geq 1$. If there exists a linear $\mathbb{Z}$-grading of Q such that f is homogeneous of degree one, then every non-zero fiber of f is rigid.*

Proof. Given $\lambda \in \mathbf{k}^*$, define the ideal $I = (f - \lambda)$ and $B = Q/I$. Let $D \in \mathrm{LND}(B)$ be given.

Assume that $\mathfrak{g}$ is a linear $\mathbb{Z}$-grading of Q relative to which $\deg f = 1$. Relative to the triple $(Q, \mathfrak{g}, I)$, we have

$$B^H = Q[w]/I^H = Q[w]/(f - \lambda w) \cong_Q Q$$

and

$$0 = D^H(f - \lambda w) = D^H(f) \ .$$

If $D \neq 0$, this equality implies $|f|_Q = 0$, contradicting the hypothesis that $|f|_Q \geq 1$. Therefore, $D = 0$, and B is rigid. □

Again, this result can be generalized by replacing $Q = \mathbf{k}^{[n]}$ with $Q = A^{[n]}$, where A is a $\mathbf{k}$-domain.

More generally, given any $f \in B$, we ask whether the natural map

$$\mathrm{LND}_f(B) \to \mathrm{LND}(B/fB)$$

is surjective.

Remark 10.2. Suppose that the $\mathbf{k}$-algebra R has a $\mathbb{Z}$-grading $\mathfrak{g}$ defined on it, and let $f \in R$ be homogeneous of degree ϕ. Given $n \geq 2$ not dividing ϕ, set $d = \gcd(\phi, n)$, and let ϕ', n' be integers such that $\phi = d\phi'$ and $n = dn'$. Define the ring

$$B = R[z]/(f + z^n) \ .$$

When $d = 1$, the $\mathbb{Z}$-grading of R induces the $\mathbb{Z}_n$-grading of B over R which we have studied; in this case, the situation for B is explicitly described in *Cor. 4.2*. When $d \geq 2$, there is no longer an induced $\mathbb{Z}_n$-grading of B over R. However, B has the $\mathbb{Z}$-grading $\mathfrak{h} = (n'\mathfrak{g}, \phi')$ defined on it, relative to which $\deg f = \deg z^n = n'\phi = n\phi'$. Thus, $\mathfrak{h}$ induces a $\mathbb{Z}_{n'}$-grading of B over R, where $n' \geq 2$. What can be said about the locally nilpotent derivations of B which are $\mathbb{Z}_{n'}$-homogeneous?

Remark 10.3. We have focused on rings $R[z]$ such that $z^n \in R$. There are three related classes of rings which naturally suggest themselves for similar investigation.

1. Rings of the form $R[z]$ where z is integral over R.
2. Rings of the form $R[z]$ where $z \in \mathrm{frac}(R)$.
3. Rings of the form $R[x, y]$ where $xy \in R$.

Note that a ring of the third type is the extended Rees algebra of a principal ideal of R.

We have the following special case for rings of the second type. This result is the counterpart to *Lemma 4.1* for rings of this type.

Lemma 10.2. *Let R be a commutative $\mathbf{k}$-domain. Suppose $a, b \in R$ are non-zero, and let z be indeterminate over R. If $|a|_R = 0$, then the ring*

$$B = R[z]/(az + b)$$

is not rigid.

Proof. Assume that there exists non-zero $\delta \in \mathrm{LND}(R)$ such that $\delta a = 0$. Let $R[x] = R^{[1]}$ and extend δ to $R[x]$ by $\delta x = 0$. Define $D \in \mathrm{LND}(R[x])$ by $D = x\delta$, noting that $Dx = Da = 0$. Next, extend D to $R[x, z] = R^{[2]}$ by setting $Dz = -\delta b$. Since $D(x - a) = D(xz + b) = 0$, it follows that D induces a non-trivial locally nilpotent quotient derivation on the quotient ring $B = R[x, z]/(x - a, xz + b)$. □

Remark 10.4. One of the main ideas of this article is to develop a systematic way to study $\mathbb{G}_a$-actions which semi-commute with an action of a cyclic group. More precisely, in sections 4 and 5, we consider a cyclic group action on a variety where the quotient map is a ramified covering space of a specific form. In the first case (adjoining one root), the ramification is always total. In the second case (adjoining two roots), there are critical points of different ramification indices. It would be of interest to use the ideas developed here to study in general the case of finite abelian group actions, and find results analogous to Theorems 4.1 and 5.1 concerning kernels and local slices of homogeneous derivations.

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