A NOTE ON RELATIVE HYPERBOLICITY AND ARTIN GROUPS

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Abstract. In [12], I. Kapovich and P. Schupp showed that certain 2-dimensional Artin groups (those with all relator indices at least 7) are hyperbolic relative (in the sense of Farb) to their non-free rank 2 parabolic subgroups. This paper considers the question of relative hyperbolicity of an Artin group with regard to the geometry of the associated Deligne complex. We prove a relative version of the Milnor-Svarc Lemma which implies, in particular, that an Artin group is hyperbolic relative (in the sense of Farb) to its finite (or spherical) type parabolic subgroups if and only if its Deligne complex is a Gromov hyperbolic space. For a 2-dimensional Artin group it is known that the Deligne complex is Gromov hyperbolic precisely when the corresponding Davis complex is Gromov hyperbolic, that is, precisely when the underlying Coxeter group is a hyperbolic group.

1. Introduction

Let G denote a finitely generated group, and $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ a finite family of subgroups of G. Let Γ_S denote the Cayley graph of G with respect to a finite generating set S. We denote by $\Gamma_{S,\mathcal{H}}$ the *coned-off Cayley graph* with respect to $\mathcal{H}$. Namely, $\Gamma_{S,\mathcal{H}}$ is the graph obtained from Γ_S by introducing a vertex V_{gH} for each left coset gH, with $g \in G$ and $H \in \mathcal{H}$, and attaching V_{gH} by an edge of length $\frac{1}{2}$ to each vertex of Γ_S labelled by an element of the coset gH. The isometric (left) action of G on the Cayley graph Γ_S clearly extends to an isometric action of G on $\Gamma_{S,\mathcal{H}}$ (by setting $g(V_{g'H}) = V_{gg'H}$ for all $g, g' \in G$ and $H \in \mathcal{H}$). The nontrivial vertex stabilizers of this action are all conjugate to subgroups of $\mathcal{H}$.

The group G is said to be *hyperbolic relative to* $\mathcal{H}$ *(in the sense of Farb)* if the coned-off Cayley graph $\Gamma_{S,\mathcal{H}}$ is a Gromov hyperbolic space (see [9]). We note that this definition can be shown to be independent of the choice of finite generating set S. (We also refer the reader to [10, 11] or other standard references, such as [5], for the definition of a Gromov hyperbolic space).

This notion of relative hyperbolicity is rather more general than the notion first conceived by Gromov [10, 11], and explored by Bowditch [4], Yaman [16], and others. The latter property is equivalent to relative hyperbolicity in the above sense together with the additional condition of *bounded coset penetration* (BCP) introduced by Farb in [9]. The stronger notion of relative hyperbolicity models more precisely certain classical situation such as that of a geometrically finite Kleinian group (which is hyperbolic relative to it parabolic subgroups in this stronger sense).

In this paper, following [12], we consider relative hyperbolicity in relation to Artin groups. As observed in [12], the presence of numerous mutually intersecting free abelian subgroups in a typical Artin group tends to preclude the possibility of relative hyperbolicity in the stronger sense. This can be made more precise (the following statement may be found in [2] and also follows without too much difficulty from Lemma 4 of [1]): if an Artin group G is relatively hyperbolic in the usual strong sense then any freely indecomposable free factor of G lies in a peripheral subgroup. In particular,

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a freely indecomposable Artin group is not relatively hyperbolic in this sense with respect to *any* collection of proper subgroups. For this reason we shall not introduce the notion of BCP [9] or any of the equivalent stronger conditions [4, 16], and shall consider simply the weaker notion defined above (which we shall indicate by the words "in the sense of Farb"). The main technical result of this paper is a "relative" version of the Milnor-Svarc Lemma (see Section 3) which states that whenever a finitely generated group G can be shown to act cocompactly, discontinuously (i.e. with discrete orbits), and isometrically on a Gromov hyperbolic length space then G is hyperbolic relative (in the sense of Farb) to a finite collection of maximal isotropy subgroups of the given action. Before turning to the proof of such a statement we first describe how it may be applied in the case of an Artin group.

2. RELATIVE HYPERBOLICITY AND ARTIN GROUPS

Let Δ denote a simplicial graph with vertex set $V(\Delta)$ and edge set $E(\Delta) \subset V(\Delta) \times V(\Delta)$. Suppose also that every edge $e = \{s,t\} \in E(\Delta)$ carries a label $m_e = m_{st} \in \mathbb{N}_{\geq 2}$. We define the *Artin group* $G(\Delta)$ associated to the (labelled) *defining graph* Δ to be the group given by the presentation [1]

$$G(\Delta) = \langle\ V(\Delta)\ \mid\ \underbrace{ststs\cdots}_{m_{st}} = \underbrace{tstst\cdots}_{m_{st}} \quad \text{for all } \{s,t\} \in E(\Delta)\ \rangle\,.$$

Adding the relations $s^2 = 1$ for each $s \in V(\Delta)$ yields a presentation of the associated *Coxeter group* $W(\Delta)$ of type Δ. We denote $\rho_\Delta : G(\Delta) \to W(\Delta)$ the canonical quotient map obtained by this addition of relations. An Artin group is said to be of *finite type* (sometimes written *spherical type*) if the associated Coxeter group is finite, and of *infinite type* otherwise. By a *standard parabolic subgroup* of $G(\Delta)$, or $W(\Delta)$, we mean any subgroup generated by a (possibly empty) subset of the standard generating set $V(\Delta)$. More generally, any subgroup which is conjugate to a standard parabolic subgroup (of $G(\Delta)$ or $W(\Delta)$) shall be referred to as a *parabolic subgroup.*

Probably the most important tool currently used in the study of *infinite* type Artin groups is the Deligne complex (see [7], etc..). We described this complex in detail.

Consider $G = G(\Delta)$ for a fixed defining graph Δ. For each subset of the generating set, $R \subset V(\Delta)$, we shall write Δ_R for the full labelled subgraph of Δ spanned by R. (Here we attach a meaning to the *empty* defining graph $\Delta_\emptyset$ by setting $G(\Delta_\emptyset) = 1$ and $W(\Delta_\emptyset) = 1$). The inclusion of Δ_R in Δ induces a homomorphism $\phi_R : G(\Delta_R) \to G$ with image the standard parabolic subgroup $\langle R\rangle$ generated by R. The construction of the Deligne complex is based on the rather nontrivial fact, due to H. van der Lek [13], that, for every defining graph Δ and every $R \subset V(\Delta)$, the homomorphism ϕ_R is an isomorphism onto its image. Thus each standard parabolic subgroup of an Artin group is itself canonically isomorphic to an Artin group. The corresponding statement for Coxeter groups is also true, and well-known (see [3]).

Define

$$\mathcal{V}_f \ = \ \{R \subset V(\Delta)\ :\ W(\Delta_R) \text{ finite }\}\,.$$

We view $\mathcal{V}_f$ as a partially ordered set under inclusion of sets, and define K to be the geometric realisation of the derived complex of $\mathcal{V}_f$. Thus there is a simplex $\sigma \in K$ of dimension $n \geq 0$ for every chain $R_0 \subset R_1 \subset \cdots \subset R_n$ of $n+1$ distinct elements in $\mathcal{V}_f$. We denote $min(\sigma) = R_0$, the minimal vertex of σ.

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[1] Our notion of defining graph differs from the frequently used "Coxeter graph" where, by contrast, the absence of an edge between s and t indicates a commuting relation ($m_{st} = 2$) and the label $m_{st} = \infty$ is used to designate the absence of a relation between s and t. In our convention the label ∞ is never used.

Note that, for $\emptyset \subseteq R \subset T \subseteq V(\Delta)$, the inclusion $\Delta_R \subset \Delta_T$ induces a homomorphism $\phi_{R,T} : G(\Delta_R) \to G(\Delta_T)$. It follows that setting $G(\sigma) = G(\Delta_{min(\sigma)})$ and $\phi_{\sigma,\tau} = \phi_{min(\sigma),min(\tau)}$, for all $\tau \subset \sigma \in K$, defines a *simple complex of groups* structure

$$(K, \{G(\cdot)\}, \{\phi_{\cdot,\cdot}\})$$

in the sense of [5]. It is easily seen that the (orbifold) *fundamental group* of this complex of groups is isomorphic to the Artin group G (via the homomorphisms $\phi_R : G(\Delta_R) \to G$), and the result of van der Lek cited above ensures that the complex of groups is *developable.* It follows that the Artin group G acts, with quotient K and isotropy subgroups the finite type parabolic subgroups $\{ G(\sigma) : \sigma \in K \}$ on a simply connected simplicial complex, the *universal cover* of $(K, \{G(\cdot)\}, \{\phi_{\cdot,\cdot}\})$, which we shall denote $\mathbb{D}$ and refer to as the *Deligne complex* associated to $G(\Delta)$.

We note that, replacing the collection of groups $\{ G(\sigma) : \sigma \in K \}$ with the corresponding Coxeter groups $\{ W(\sigma) : \sigma \in K \}$ leads in a similar way to the definition of a developable complex of groups whose fundamental group is, this time, the Coxeter group $W = W(\Delta)$, and whose universal cover, which we shall denote $\mathbb{D}_W$, is known as the *Davis complex.* The Coxeter group acts on its Davis complex with finite vertex stabilizers (in fact properly discontinuously and cocompactly). This is quite different to the situation of the Deligne complex where the Artin group acts with every nontrivial vertex stabilizer an infinite group. In particular, the Deligne complex is not even a locally compact space (while the Davis complex clearly is). Nevertheless, a lot of important information is carried by the action of the Artin group on its Deligne complex, as can be seen from [7, 8] etc.

We suppose now that the complex K is endowed with a piecewise Euclidean metric. Since K is finite, this induces a complete G-equivariant length metric on the Deligne complex (c.f. [5]). There are two very natural choices for such a metric (namely the Moussong metric and the cubical metric) described in [7]. These specific metrics are particularly useful when they can be shown to be nonpositively curved, or CAT(0), as demonstrated in [7], where we refer the reader for further details. However, in what follows, the actual choice of piecewise Euclidean metric is more or less irrelevant since for any such metric d, the Deligne complex $(\mathbb{D}, d)$ will be (G-equivariantly) quasi-isometric to the 1-skeleton of $\mathbb{D}$ equipped with the unit-length edge metric. The statement "the Deligne complex is Gromov hyperbolic" shall henceforth be interpreted to mean "'with respect to any piecewise Euclidean metric".

The main result of this paper is the following:

Theorem 2.1. *An Artin group is hyperbolic relative (in the sense of Farb) to its finite type standard parabolic subgroups if and only if its Deligne complex is a Gromov hyperbolic space.*

Proof. It is easily seen that the action of an Artin group on its Deligne complex (equipped with a piecewise Euclidean length metric) is discontinuous and co-compact. (As observed later, in Remark 3.2, this is actually true in the case of an arbitrary finite developable complex of groups). Also, the (maximal) isotropy subgroups of this action are, by construction, just the (maximal) finite type parabolic subgroups of the Artin group. The result now follows immediately from the relative version of the Milnor-Svarc Lemma (Theorem 3.1) proved in the Section 3. ☐

In [12], I. Kapovich and P. Schupp showed that an Artin group with all indices $m_{ij} \geq 7$ is hyperbolic relative (in the sense of Farb) to its finite type parabolic subgroups. For these examples, finite type parabolic means simply rank 2 and non-free.

More generally, we say that the Artin group $G = G(\Delta)$ is *2-dimensional* if Δ has at least one edge (G is not free) and every triangle in Δ has edge labels m, n, p satisfying $1/m + 1/n + 1/p \leq 1$, equivalently if every rank 3 parabolic subgroup is of infinite type. The terminology is justified by the fact that an Artin group is 2-dimensional in this sense if and only if it has cohomological dimension 2. Each 2-dimensional Artin group is also known to have geometric dimension 2. Moreover, the Deligne complex is 2-dimensional and is CAT(0) when equipped with the Moussong metric. These statements were all established in the paper of Charney and Davis [7].

It is clear that the groups treated by Kapovich and Schupp are all examples of 2-dimensional Artin groups. The following statement generalises their result.

Corollary 2.2. *Let $G(\Delta)$ be a 2-dimensional Artin group. Then the following are equivalent*

1. *the Coxeter group $W(\Delta)$ is a hyperbolic group;*
2. *the Deligne complex $\mathbb{D}$ (equipped with the Moussong metric) is Gromov hyperbolic;*
3. *Δ contains no triangle having edge labels m, n, p with $1/m + 1/n + 1/p = 1$ and no square with all edge labels equal to 2.*
4. *$G(\Delta)$ is hyperbolic relative (in the sense of Farb) to the collection of parabolic subgroups $\{G(e) \,:\, e \in E(\Delta)\}$.*

Proof. The equivalence of (1)–(3) is proved in [8] Lemma 5 (using the Flat Plane Theorem and the relationship between the Deligne complex and the Davis complex). The equivalence of (2) and (4) is a consequence of the above Theorem. ☐

We observe that there is a close relationship between the Deligne complex $\mathbb{D}$ associated to an Artin group $G = G(\Delta)$ and the Davis complex $\mathbb{D}_W$ for the corresponding Coxeter group $W = W(\Delta)$. The canonical projection induces a simplicial map $\mathbb{D} \to \mathbb{D}_W$ which is surjective, and the Tits section (a setwise section to the canonical projection) induces an inclusion $\mathbb{D}_W \hookrightarrow \mathbb{D}$. We suppose that $\mathbb{D}_W$ is equipped with an equivariant metric. Since the Coxeter group W acts properly co-compactly and isometrically on $\mathbb{D}_W$ the space and the group are quasi-isometric (by the Milnor-Svarc Lemma). Thus W is a hyperbolic group precisely when $\mathbb{D}_W$ is a Gromov hyperbolic space. On the other hand, by Theorem 2.1, relative hyperbolicity of the Artin group is equivalent to Gromov hyperbolicity of the Deligne complex. The form of the above Corollary (covering the 2-dimensional case) now suggests the following conjecture.

Conjecture 2.3. *The Deligne complex $\mathbb{D}$ associated to an Artin group is Gromov hyperbolic if and only if the corresponding Davis complex $\mathbb{D}_W$ is Gromov hyperbolic.*

We note that, by the work of Moussong [15], the Davis complex equipped with the Moussong metric $(\mathbb{D}_W, d_M)$ is, in all cases, a CAT(0) space and is Gromov hyperbolic if and only if it contains no isometrically embedded flat plane. It is conjectured in [7] that the Moussong metric *on the Deligne complex* is always CAT(0). One can check easily enough that whenever this is indeed the case (notably in the 2-dimensional case) then the Conjecture 2.3 also holds. This allows one to extend slightly the statement of Corollary 2.2 to include certain examples considered by Charney in [6] (see Corollaries 5.5 and 5.6) which allow 3-dimensional parabolic subgroups isomorphic to the 4-string braid group.

One further case where this conjecture might reasonably be tackled is that of an Artin group of type FC (see [7] for a definition). An Artin group is of type FC precisely when the cubical metric on the Deligne complex (defined in [7]) is CAT(0). In this case it may be checked that the inclusion of the Davis complex into the Deligne complex (induced by the Tits section) is an isometric embedding with respect to the CAT(0) cubical metrics. It follows (by the Flat Plane Theorem) that, for an FC type Artin group, the Deligne complex is Gromov hyperbolic *only if* the Davis complex is. The reverse implication seems to be rather more difficult.

3. A RELATIVE VERSION OF THE MILNOR-SVARC LEMMA

Let X be a metric space and G a group which acts on X by isometries. We say that the action is *co-compact* if there exists a compact set $K \subset X$ such that $X = \bigcup_{g \in G} gK$, and *discontinuous* if every orbit is a discrete subspace of X, equivalently if for all $x \in X$ there exists an $\epsilon_x > 0$ such that $d(x, y) > \epsilon_x$ for all $y \in G(x) \setminus \{x\}$. A subgroup $H < G$ is said to be an *isotropy subgroup* of G if its fixed set in X is non-empty.

Theorem 3.1. *Let G be a finitely generated group and suppose that G admits a discontinuous, co-compact, isometric action on a length space X. Let $\mathcal{H}$ denote a collection of subgroups of G consisting of exactly one representative of each conjugacy class of maximal isotropy subgroups for the action of G on X. Then $\mathcal{H}$ is finite and, for any finite generating set S of G, the coned-off Cayley graph $\Gamma_{S,\mathcal{H}}(G)$ is quasi-isometric to X. In particular, if X is a Gromov hyperbolic space then the group G is hyperbolic relative (in the sense of Farb) to the collection $\mathcal{H}$ of maximal isotropy subgroups.*

Proof. We first show that the number of conjugacy classes of maximal isotropy subgroups is finite. Clearly, distinct maximal isotropy subgroups have disjoint fixed sets in X. Also, since G acts co-compactly, every maximal isotropy subgroup is conjugate to one which fixes a point inside a certain compact region K such that $X = \bigcup_{g \in G} gK$. By way of contradiction, we now suppose that there exists an infinite sequence $H_1, H_2, \dots$ of pairwise distinct maximal isotropy subgroups such that each H_i fixes a point $x_i \in K$ (the points x_i being necessarily pairwise distinct). By compactness of K we may pass to an infinite subsequence for which the sequence $(x_i)_{i \in \mathbb{N}}$ converges to a point $x \in K$. Moreover, at most one of the H_i may fix x, so we may as well suppose that none of them fix x. For each i we may therefore choose an element $h_i \in H_i$ such that $h_i(x) \neq x$. Since $d(h_i(x), x) \leq 2d(x_i, x)$ it follows that the sequence $(h_i(x))_{i \in \mathbb{N}}$ also converges to x, contradicting the assumption that G acts discontinuously.

Let $H_1, H_2, \dots, H_n$ denote the finitely many maximal isotropy subgroups whose fixed sets intersect K nontrivially. Set $Q = \{g \in G : gK \cap K \neq \emptyset \text{ but } g \notin H_r \text{ for all } r = 1, \dots, n\}$, and choose a subset $\widehat{Q} \subset Q$ which contains exactly one representative for each coset gH_r, for $g \in Q$ and $r \in \{1, \dots, n\}$. We note that G is generated by the set $\widehat{Q}$ together with the subgroups $H_1, \dots, H_n$.

We now use a compactness argument to show that $\widehat{Q}$ is finite. By way of contradiction we suppose that there exists an infinite sequence $(g_i)_{i \in \mathbb{N}}$ of pairwise distinct elements of $\widehat{Q}$. For each $i \in \mathbb{N}$ we may find $x_i, y_i \in K$ such that $g_i(x_i) = y_i \in gK \cap K$. Since $K \times K$ is compact we may pass to an infinite subsequence of $(g_i)_{i \in \mathbb{N}}$ for which the sequence of pairs (x_i, y_i) converges to a point $(x, y) \in K \times K$. Since

$$d(g_i(x), y) \leq d(g_i(x), g_i(x_i)) + d(g_i(x_i), y) = d(x, x_i) + d(y_i, y)$$

it follows that the sequence $(g_i(x))_{i \in \mathbb{N}}$ converges to y. Since the action of G is discontinuous, this implies that the sequence is eventually constant: there exists N such that $g_i(x) = y$ for all $i > N$. But then, for any $N < i < j$, the element $g_j^{-1} g_i$ fixes $x \in K$ and hence lies in some maximal isotropy subgroup H_r. That is to say that g_i and g_j are different representatives in $\widehat{Q}$ for the same coset of H_r, contradicting the choice of $\widehat{Q}$.

Since G is finitely generated we may extract a finite generating set from any given generating set for the group. It follows that we may extend $\widehat{Q}$ to a finite generating set S of G in such a way that $\widehat{Q} \subset S \subset \widehat{Q} \cup H_1 \cup \dots \cup H_n$. Let $\Gamma = \Gamma_{S,\mathcal{H}}(G)$ denote the coned-off Cayley graph for G with respect to the generating set S and a finite set $\mathcal{H}$ of isotropy subgroups as prescribed in the statement of the Theorem. We may as well suppose that $\mathcal{H}$ is chosen by selecting from amongst the subgroups H_r, $r = 1, \dots, n$, one from each conjugacy class. Moreover, the exact choice of representatives for $\mathcal{H}$ is not really important, as the structure of the coned-off Cayley graph Γ depends only on the set of conjugacy classes of subgroups involved. Note also that, up to quasi-isometry, Γ depends only on the set $\widehat{Q}$ (rather than the choice of S) and the subgroups H_r. More generally, the coned-off Cayley graph $\Gamma = \Gamma_{S,\mathcal{H}}(G)$ is independent, up to quasi-isometry, of the choice of generating set S (regardless of whether or not it contains $\widehat{Q}$) as long as this set is finite.

Let v_0 denote the base vertex of Γ and write Γ_0 for the G-orbit of v_0 with metric induced from Γ. Then the inclusion $\Gamma_0 \to \Gamma$ is a quasi-isometry. We shall show that Γ_0 (and therefore Γ) is quasi-isometric to X.

Choose a point $x_0 \in K$. This choice determines a G-equivariant map $f : \Gamma_0 \to X$ by sending gv_0 to gx_0 for all $g \in G$. It is easily seen that, for $p, q \in \Gamma_0$,

$$d_X(f(p), f(q)) \le R\, d_\Gamma(p, q)\,,$$

where R denotes the maximum value in the finite set

$$\{\, d_X(x_0, s(x_0)) \;:\; s \in \widehat{S}\,\} \cup \{\, 2d_X(x_0, Fix(H_r)) \;:\; r = 1, \dots, n\,\}\,.$$

In order to prove the reverse inequality (i.e. to bound $d_\Gamma(p, q)$ above by a linear function of $d_X(f(p), f(q))$) we need to establish the following fact:

There exists a constant $\epsilon > 0$ such that, for all $g \in G$, either $gK \cap K \neq \emptyset$ or $d_X(gK, K) > \epsilon$ (where here we understand the Hausdorff distance).

We use, once again, a compactness argument to prove this statement. If the statement is not true then we may find a sequence $(g_i)_{i\in\mathbb{N}}$ of distinct group elements such that $0 < d_X(g_{i+1}K, K) < \frac{1}{2} d_X(g_iK, K)$ for all $i \in \mathbb{N}$. Choosing, for each i, a pair $(x_i, y_i) \in K \times K$ such that $d_X(g_iK, K) < d_X(g_i(x_i), y_i) < 2d_X(g_iK, K)$, and passing to an infinite subsequence for which the sequence of pairs converges to a pair $(x, y) \in K \times K$, we observe that the sequence $(g_i(x))_{i\in\mathbb{N}}$ converges to y. This contradicts the assumption that the action of G is discontinuous unless the sequence is eventually stationary, that is, unless $g_i(x) = y$ for some i. But this is impossible since $g_iK \cap K = \emptyset$ for all $i \in \mathbb{N}$.

we shall now give an upper bound for $d_\Gamma(p, q)$. Since X is a path metric space we may find a path γ from $f(p)$ to $f(q)$ in X whose length approximates the distance between these points to within ϵ: $\ell(\gamma) \le d_X(f(p), f(q)) + \epsilon$. Choose $m \in \mathbb{N}$ such that $(m-1)\epsilon < \ell(\gamma) \le m\epsilon$, and let $f(p) = y_0, y_1, \dots, y_m = f(q)$ denote equally spaced points along the path γ. In particular $d(y_{i-1}, y_i) \le \epsilon$, for all $i = 1, \dots, m$. Choosing $K_0, K_1, K_2, \dots, K_m$ to be translates of the compact K such that $y_i \in K_i$ for all i, we observe, by the claim just proven, that $K_{i-1} \cap K_i \neq \emptyset$ for all $i = 1, \dots, m$. By construction, whenever $gK \cap K \neq \emptyset$ we may express g in the form sh for $s \in \widehat{S}$ and $h \in H_r$ (for some r). Thus the sequence $K_0, K_1, K_2, \dots, K_m$ gives rise to a path of length at most $2m$ joining p to q in Γ. Thus $d_\Gamma(p, q) \le 2m$. On the other hand $(m-1)\epsilon < \ell(\gamma) \le d_X(f(p), f(q)) + \epsilon$. Combining these inequalities results in

$$d_\Gamma(p, q) \;<\; \frac{2}{\epsilon} d_X(f(p), f(q)) + 4\,.$$

This completes the proof that the map $f : \Gamma_0 \to X$ is a quasi-isometric embedding. Clearly, since the compact K is bounded, any point in X is a bounded distance from a point in the orbit of x_0, and so the map f is in fact a quasi-isometry. ☐

Example 3.2. A rather general construction is to describe a group G as the fundamental group of a complex of groups (see [5]). If G is the fundamental group of a finite complex $(Y, \mathcal{G})$ of groups which is developable and whose universal cover is Gromov hyperbolic, then G is hyperbolic relative to $\mathcal{G}$ (in the sense of Farb). As discussed in the preceding section, each Artin group is the fundamental group of a finite complex of groups, leading to the results stated there.

Example 3.3. A further example is that of the mapping class group $Mod(S)$ of a closed orientable surface S of higher genus. Recall that the complex of curves $\mathcal{C}(S)$ associated to the closed surface S is defined to be the simplicial complex with vertex set the set of nontrivial isotopy classes of simple closed curves and a simplex spanned by each collection of vertices which is simultaneously represented by a collection of disjoint (non-parallel) simple closed curves. The group $Mod(S)$ acts naturally on this complex (with rather large vertex stabilizers). In [14], Masur and Minsky showed that the complex of curves $\mathcal{C}(S)$ is a Gromov hyperbolic space and used this to prove that the mapping class group $Mod(S)$ is relatively hyperbolic (in the sense of Farb) with respect to subgroups $H_C := \{g \in Mod(S) : g[C] = [C]\}$ for a finite collection of simple closed curves C in S. Their proof passes through a modified version of Teichmuller space ("electric space") which they show to be quasi-isometric to $\mathcal{C}(S)$. A proof of this result may also be obtained by applying Theorem 3.1 directly to the complex $\mathcal{C}(S)$. This avoids introducing the action on Teichmuller space while still invoking the hyperbolicity of the curve complex as proved in [14].

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