Adv. Geom. **5** (2005), 607–636

Advances in Geometry

Artin groups of type *B* and *D*

John Crisp and Luis Paris

(Communicated by A. Cohen)

Abstract. We show that each of the Artin groups of type B_n and D_n can be presented as a semidirect product $F \rtimes \mathscr{B}_n$, where F is a free group and $\mathscr{B}_n$ is the n-string braid group. We explain how these semidirect product structures arise quite naturally from fibrations, and observe that, in each case, the action of the braid group $\mathscr{B}_n$ on the free group F is classical. We prove that, for each of the semidirect products, the group of automorphisms which leave invariant the normal subgroup F is small: namely, $\mathrm{Out}(A(B_n), F)$ has order 2, and $\mathrm{Out}(A(D_n), F)$ has order 4 if n is even and 2 if n is odd. It is known that the Artin group of type D_n may be viewed as an index 2 subgroup of the n-string braid group over a disk with a degree 2 orbifold point. We show that this orbifold braid group has outer automorphism group of order 2, for all $n \geqslant 2$.

Key words. Braid groups, Artin groups, automorphisms.

2000 Mathematics Subject Classification. Primary 20F36; Secondary 20F28

1 Introduction

Let S be a finite set. A *Coxeter matrix* over S is a matrix $M = (m_{\alpha\beta})_{\alpha,\beta \in S}$ indexed by the elements of S such that $m_{\alpha\alpha} = 1$ for all $\alpha \in S$, and $m_{\alpha\beta} = m_{\beta\alpha} \in \{2, 3, 4, \dots, +\infty\}$ for all $\alpha, \beta \in S$, $\alpha \neq \beta$. A Coxeter matrix $M = (m_{\alpha\beta})$ is usually represented by its *Coxeter graph*, Γ. This (labelled) graph is defined as follows: the set S is identified with the set of vertices of Γ, and two vertices α, β are joined by an edge in Γ if $m_{\alpha\beta} \geqslant 3$, the edge being labelled by $m_{\alpha\beta}$ if $m_{\alpha\beta} \geqslant 4$. For $\alpha, \beta \in S$ and $m \in \mathbb{Z}_{\geqslant 2}$ we denote by $w(\alpha, \beta : m)$ the word $\alpha\beta\alpha\dots$ of length m. Define the *Artin group* of type Γ to be the (abstract) group $A(\Gamma)$ presented by

$$A(\Gamma) = \langle S \mid w(\alpha, \beta : m_{\alpha\beta}) = w(\beta, \alpha : m_{\alpha\beta}) \text{ for } \alpha \neq \beta \text{ and } m_{\alpha\beta} < +\infty \rangle.$$

The *Coxeter group* $W(\Gamma)$ of type Γ is the quotient of $A(\Gamma)$ by the relations $\alpha^2 = 1$, $\alpha \in S$. The cardinal of S is called the *rank* of $A(\Gamma)$. We say that $A(\Gamma)$ is of *spherical type* if $W(\Gamma)$ is finite, and that $A(\Gamma)$ is *irreducible* if Γ is connected. Note that, if $\Gamma_1, \Gamma_2, \dots, \Gamma_\ell$ are the connected components of Γ, then $A(\Gamma) = A(\Gamma_1) \times A(\Gamma_2) \times \dots \times A(\Gamma_\ell)$. If Γ is the graph A_{n-1} shown in Figure 1, then $A(\Gamma) = A(A_{n-1})$

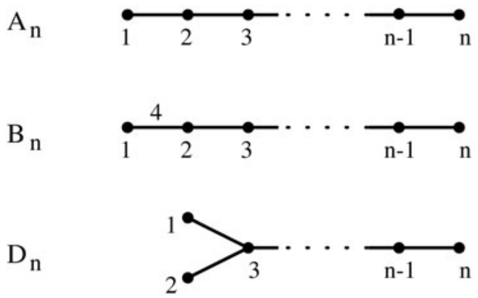

Figure 1. Coxeter graphs of type A_n, B_n and D_n.

is the Artin braid group on n strings, which we shall also denote by $\mathscr{B}_n$, and $S = \{\alpha_1, \dots, \alpha_{n-1}\}$ is the set of standard generators of $\mathscr{B}_n$. Moreover, the group $W(A_{n-1})$ is the symmetric group on $\{1, \dots, n\}$, which we shall also denote by $\mathscr{S}_n$, and the α_i correspond in this group to the transpositions $(i, i+1)$, $i = 1, \dots, n-1$.

Artin groups, also called *generalized braid groups*, were first introduced by Tits [25] as extensions of Coxeter groups. Later, Brieskorn [7] gave a topological interpretation of the Artin groups of spherical type in terms of regular orbit spaces as follows. Define a *finite reflection group* to be a finite subgroup W of $O(n, \mathbb{R})$ generated by reflections, where n is some positive integer. A classical result due to Coxeter [14], [15] states that W is a finite reflection group if and only if W is a finite Coxeter group. Assume this is the case. Let $\mathscr{A}(\Gamma)$ be the set of reflecting hyperplanes of $W = W(\Gamma)$, and, for $H \in \mathscr{A}(\Gamma)$, let $H_{\mathbb{C}}$ denote the hyperplane of $\mathbb{C}^n$ having the same equation as H. Let

$$M(\Gamma) = \mathbb{C}^n \Big\backslash \Big(\bigcup_{H \in \mathscr{A}(\Gamma)} H_{\mathbb{C}} \Big).$$

Then $M(\Gamma)$ is a connected submanifold of $\mathbb{C}^n$, the group $W(\Gamma)$ acts freely on $M(\Gamma)$, and the quotient $N(\Gamma) = M(\Gamma)/W(\Gamma)$ is isomorphic to the complement in $\mathbb{C}^n$ of an algebraic variety called *discriminant variety* of type Γ. Now, by a theorem of Brieskorn [7], the fundamental group of $N(\Gamma)$ is isomorphic to the Artin group $A(\Gamma)$. In the case $\Gamma = A_{n-1}$, $W(A_{n-1}) = \mathscr{S}_n$ acts on $\mathbb{R}^n$ by permuting the coordinates, the reflections in $\mathscr{S}_n$ are exactly the transpositions, and $N(A_{n-1})$ is the space of configurations of n (unordered) points in $\mathbb{C}$, whose fundamental group is well-known to be the braid group $\mathscr{B}_n$.

Since the work of Brieskorn and Saito [9] and that of Deligne [17], the combinatorial theory of Artin groups of spherical type has been well studied. In particular, these groups are known to be biautomatic (see [10], [11]). The finite irreducible Coxeter groups, and therefore the irreducible Artin groups of spherical type, were classified by Coxeter [15]. These consist of the three infinite families of Coxeter groups defined by the Coxeter graphs A_n, B_n and D_n of Figure 1, the dihedral groups of order $2m$ for $m \geqslant 3$, associated to the Artin groups of rank 2 (with $m_{\alpha\beta} = m_{\beta\alpha} = m \neq +\infty$), as well as the 6 so-called sporadic reflection groups.

In this paper, for each of the Artin groups of type B_n and D_n we present semidirect product structures $F \rtimes \mathscr{B}_n$, where F is a free group and $\mathscr{B}_n$ is the n-string braid group. We explain how these structures arise quite naturally from fibrations of the regular orbits spaces $N(B_n)$ and $N(D_n)$ based on $N(A_{n-1})$. Moreover, we observe that, in each case, the action of the braid group $\mathscr{B}_n$ on the free group F is classical: for type B_n it is Artin's representation [2, 4], and for type D_n it comes from the monodromy action on the Milnor fibre of the singularity of type A_{n-1} (see Perron–Vannier [24]).

The automorphism groups of the (spherical type) Artin groups are just beginning to be explored. Artin's 1947 paper [3] was motivated by the problem of determining the automorphism groups of the braid groups (it is explicit in the introduction). However, the problem itself was solved only 37 years later by Dyer and Grossman [18] who proved that the outer automorphism group of the braid group $\mathscr{B}_n$ is of order 2 generated by the automorphism which sends each standard generator to its inverse. Until now, except for the braid groups, the only known significant result on the automorphism groups of (spherical type) Artin groups is an extension of Artin's results of [3] to all irreducible Artin groups of spherical type (see [13]). In this paper we prove that, for each of the above semidirect products, the group of automorphisms which leave invariant the normal subgroup F is small: namely $\mathrm{Out}(A(B_n), F)$ has order 2 (see Theorem 4.2), and $\mathrm{Out}(A(D_n), F)$ has order 4 if n is even, and 2 if n is odd (see Theorem 4.9). It is an open question as to whether, in either the B_n or the D_n case, the free subgroup F is a characteristic subgroup of the Artin group.

The Artin group of type D_n may be viewed as an index 2 subgroup of the n-string braid group over an orbifold (namely the complex plane with a single orbifold point of degree 2, see Allcock [1]). This latter group can be presented as a semidirect product $K \rtimes \mathscr{B}_n$, where K is the free product of n copies of $C_2 = \{\pm 1\}$, and it shall play a major role in our study of the Artin groups of type D_n. We also show in the paper that the outer automorphism group of this group is of order 2.

Finally we remark that the automorphism groups of the spherical type Artin groups of rank 2 are known, by work of N. D. Gilbert, J. Howie, V. Metaftsis and E. Raptis [20]. These groups have presentation $A = \langle a, b \mid w(a, b : m) = w(b, a : m) \rangle$, where $m \geqslant 2$. The outer automorphism group $\mathrm{Out}(A)$ is of order 2 if m is odd (Theorem C [20]) and is isomorphic to $(\mathbb{Z} \rtimes C_2) \times C_2$ if m is even (Theorem D [20]). In particular, this shows that the automorphism group of a spherical type Artin group need not be finite.

2 The semidirect product structures

We number the vertices of the Coxeter graphs of type A_n, B_n and D_n as shown in Figure 1. The standard generators of $A(A_n)$ will be written $\alpha_1, \alpha_2, \dots, \alpha_n$, the standard generators of $A(B_n)$ will be written $\beta_1, \beta_2, \dots, \beta_n$, and the standard generators of $A(D_n)$ will be written $\delta_1, \delta_2, \dots, \delta_n$. For the rest of the article we shall identify $A(A_{n-1})$ with the classical n-string braid group $\mathscr{B}_n$ in the usual fashion. This will simplify notation throughout, and be a constant reminder of the special role played by the braid group $\mathscr{B}_n$ in this story.

2.1 Definition of π_B, π_D, s_B, and s_D. We wish to study the epimorphisms $\pi_B : A(B_n) \to \mathcal{B}_n$ and $\pi_D : A(D_n) \to \mathcal{B}_n$ defined by

$$\pi_B(\beta_1) = 1, \quad \pi_B(\beta_i) = \alpha_{i-1} \quad \text{for } i = 2, 3, \dots, n.$$
$$\pi_D(\delta_1) = \pi_D(\delta_2) = \alpha_1, \quad \pi_D(\delta_i) = \alpha_{i-1} \quad \text{for } i = 3, \dots, n.$$

The epimorphism $\pi_B : A(B_n) \to \mathcal{B}_n$ admits a section $s_B : \mathcal{B}_n \to A(B_n)$ defined by

$$s_B(\alpha_i) = \beta_{i+1} \quad \text{for } i = 1, 2, \dots, n-1.$$

In particular, $A(B_n)$ may be written as a semidirect product $A(B_n) = \ker \pi_B \rtimes \mathcal{B}_n$.

In a similar fashion, the epimorphism $\pi_D : A(D_n) \to \mathcal{B}_n$ admits a section $s_D : \mathcal{B}_n \to A(D_n)$ defined by

$$s_D(\alpha_i) = \delta_{i+1} \quad \text{for } i = 1, 2, \dots, n-1.$$

and $A(D_n)$ may be written as a semidirect product $A(D_n) = \ker \pi_D \rtimes \mathcal{B}_n$.

We now describe how each of these semidirect products arise from a topologically defined faithful action of the braid group $\mathcal{B}_n$ on a free group. Both of the representations involved are classical. The first is quite famous and due to Artin [2]. The second arises from the monodromy action of the n-string braid group on the Milnor fibre of the singularity of type A_{n-1}, and was shown to be faithful by Perron and Vannier [24].

We recall the definition of a *Dehn twist* homeomorphism $\tau_c : \Sigma \to \Sigma$ on a simple closed curve c in a surface Σ. Let A denote an annular neighbourhood of c. Then τ_c is any homeomorphism isotopic to one which is the identity on $\Sigma \setminus \mathrm{int}(A)$ and transforms the interior of A as shown in Figure 2. If c bounds a disk in Σ and a, b are points on c which are interchanged by τ_c, then τ_c induces what we shall call a *braid twist* homeomorphism σ_c of the punctured surface $\Sigma \setminus \{a, b\}$. This homeomorphism exchanges (neighbourhoods of) the two punctures, as shown in Figure 2.

2.2 Artin's representation. Let Σ_B denote the surface of Figure 3, namely $\mathbb{C} \setminus \{1, 2, 3, \dots, n\}$, and, for each $i = 1, \dots, n-1$, let σ_i denote the braid twist homeomorphism on the curve c_i illustrated in Figure 3. Take the origin $0 \in \mathbb{C}$ as a basepoint and define loops γ_i at 0 for $i = 1, \dots, n$, as shown in the figure. Then $\pi_1(\Sigma_B, 0) = F_n$ is freely generated by the elements $u_i = [\gamma_i]$. Artin's representation $\rho_B : \mathcal{B}_n \to \mathrm{Aut}(F_n)$ is defined by $\rho_B(\alpha_i) = (\sigma_i)_*$, for $i = 1, 2, \dots, n-1$.

Proposition 2.1. (1) *Let F_n denote the free group on n generators $u_1, u_2, \dots, u_n$. Then Artin's representation $\rho_B : \mathcal{B}_n \to \mathrm{Aut}(F_n)$ is well-defined and given algebraically by*

$$\rho_B(\alpha_i) : \begin{cases} u_i \mapsto u_{i+1} \\ u_{i+1} \mapsto u_{i+1}^{-1} u_i u_{i+1} \\ u_j \mapsto u_j \quad j \notin \{i, i+1\}. \end{cases}$$

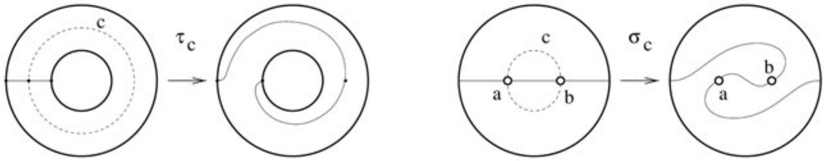

Figure 2. Dehn twist τ_c, and braid twist σ_c.

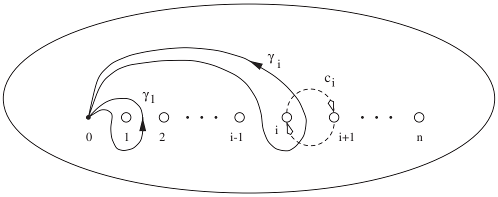

Figure 3. The surface $\Sigma_B = \mathbb{C}\backslash\{1, 2, \dots, n\}$, free group generators $u_i = [\gamma_i]$, and braid twists $\sigma_i = \rho_B(\alpha_i)$ about c_i.

(2) *We have* $A(B_n) \cong F_n \rtimes_{\rho_B} \mathscr{B}_n$, *where the projection onto the second factor is* π_B *and the section* $\mathscr{B}_n \hookrightarrow F_n \rtimes \mathscr{B}_n$ *is just* s_B. *In particular*, $\ker \pi_B$ *is a free group of rank* n.

Proof. Part (1) is well-known and due to Artin [2]. It is easily enough checked that this algebraic formulation gives a well-defined representation and corresponds to the topological description given above after identifying F_n with $\pi_1 \Sigma_B$ via $u_i = [\gamma_i]$ for $i = 1, 2, \dots, n$. We concern ourselves with Part (2) of the proposition.

Let $\varphi : \{\beta_1, \dots, \beta_n\} \to F_n \rtimes \mathscr{B}_n$ be the function defined by $\varphi(\beta_1) = u_1$ and $\varphi(\beta_i) = \alpha_{i-1}$ for $i = 2, \dots, n$. It is easily verified that φ extends to a homomorphism $\varphi : A(B_n) \to F_n \rtimes \mathscr{B}_n$.

Let $\psi : \{u_1, \dots, u_n\} \cup \{\alpha_1, \dots, \alpha_{n-1}\} \to A(B_n)$ be defined by

$$\psi(u_i) = \beta_i \beta_{i-1} \dots \beta_2 \beta_1 \beta_2^{-1} \dots \beta_{i-1}^{-1} \beta_i^{-1} \quad \text{for } i = 1, \dots, n.$$

$$\psi(\alpha_i) = \beta_{i+1} \quad \text{for } i = 1, \dots, n-1.$$

Again, it is easily seen that ψ induces a homomorphism $\psi : F_n \rtimes \mathscr{B}_n \to A(B_n)$. (Hint: verify the relation $\psi(\alpha_i u_i u_{i+1} \alpha_i^{-1}) = \psi(u_i u_{i+1})$, for all $1 \leqslant i < n$, by induction on i). Finally, one checks that $\varphi \circ \psi = \mathrm{id}$ and $\psi \circ \varphi = \mathrm{id}$. ☐

Proposition 2.2 (Artin [2, 4], Magnus [22]). *The representation* $\rho_B : \mathscr{B}_n \to \mathrm{Aut}(F_n)$ *is faithful and its image is the set of automorphisms that permute the conjugacy classes of the elements* $u_1, u_2, \dots, u_n$ *and fix the product* $u_1 u_2 \dots u_n$.

2.3 Braid monodromy representation. Let Σ_D denote the surface shown in Figure 4 and, for $i = 1, \dots, n-1$, let τ_i denote the Dehn twist homeomorphism on the curve c_i illustrated in the figure. Choose a basepoint p and loops γ_i at p for $i = 1, \dots, n-1$,

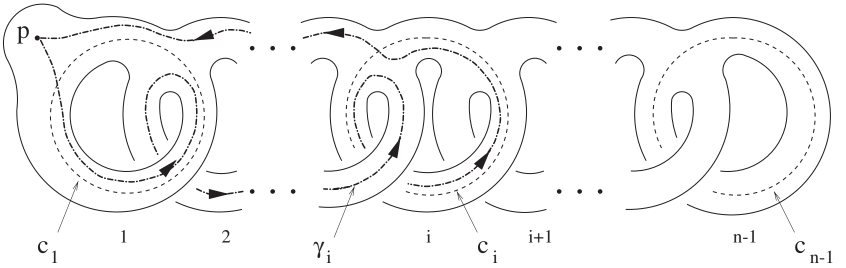

Figure 4. The surface Σ_D, free group generators $v_i = [\gamma_i]$, and Dehn twists $\tau_i = \rho_D(\alpha_i)$ about c_i.

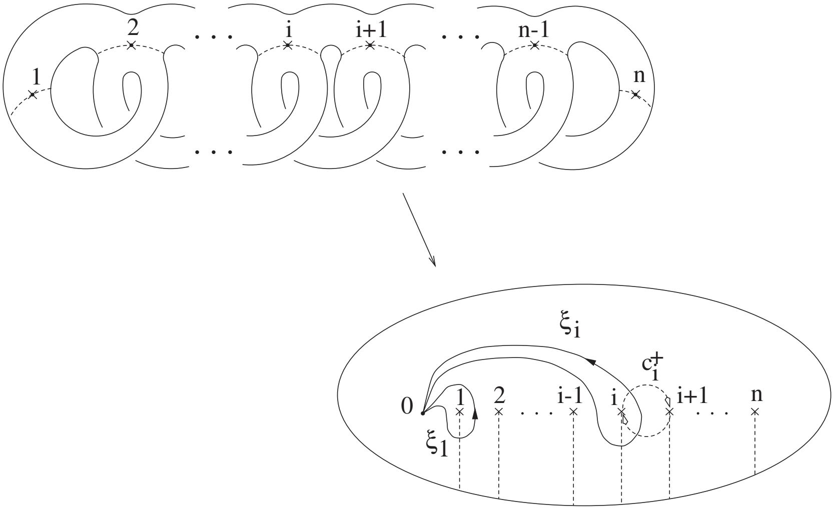

Figure 5. The twofold branched cover $\Sigma_D \to \Sigma^+$, generators $x_i = [\xi_i]$ for $*_n(C_2)$, and braid twists $\sigma_i^+ = \rho^+(\alpha_i)$ about c_i^+.

as shown in the figure. Then $\pi_1(\Sigma_D, p) = F_{n-1}$ is freely generated by the elements $v_i = [\gamma_i]$. The braid monodromy representation $\rho_D : \mathscr{B}_n \to \mathrm{Aut}(F_{n-1})$ is defined by $\rho_D(\alpha_i) = (\tau_i)_*$, for $i = 1, 2, \ldots, n-1$.

For what follows in the later sections, it is important to view the surface Σ_D as a branched 2-fold cover of $\mathbb{C}$ branched over the set $\{1, 2, \ldots, n\}$, as shown in Figure 5. Note that the quotient orbifold Σ^+ has orbifold fundamental group $*_n(C_2)$ (the free product of n groups of order 2). Let $\kappa : *_n(C_2) \to C_2$ be the epimorphism which maps each free factor nontrivially. Note that κ is the unique epimorphism $*_n(C_2) \to C_2$ which sends every element of order 2 to the generator of C_2, hence the kernel of κ is a characteristic subgroup. Furthermore, $\ker\kappa = F_{n-1}$ is free of rank $n-1$ and the surface Σ_D is the branched cover of Σ^+ corresponding to $F_{n-1} = \ker\kappa < *_n(C_2)$.

We suppose that the basepoint p in Σ_D maps onto the origin $0 \in \Sigma^+$. The loops ξ_i at 0, for $i = 1, \ldots, n$, as shown in Figure 5, represent canonical generators $x_1, x_2, \ldots, x_n$

for the group $*_n(C_2)$. The loops γ_i were chosen to be (homotopic to) lifts of the loops $\xi_1\xi_{i+1}$, for $i = 1, \dots, n-1$, and represent one choice of generators for the free subgroup F_{n-1}. A different choice will be used in a later section. The Dehn twist along the curve c_i is a lift of the braid twist along the curve c_i^+ in Σ^+. Thus ρ_D extends to a representation $\rho^+ : \mathscr{B}_n \to \mathrm{Aut}(*_n C_2)$ which is induced from ρ_B under the map $F_n \to *_n C_2 = F_n/(u_i^2, i = 1, \dots, n)$.

The following is simply a more explicit version of Perron and Vannier [24], Corollaire 1, where it is observed that $A(D_n)$ is a semidirect product of a rank $n-1$ free group and the n-string braid group.

Proposition 2.3. *Suppose that* $n \geqslant 4$.

(1) *Let* F_{n-1} *denote the free group on* $n-1$ *generators* $v_1, v_2, \dots, v_{n-1}$. *Then the braid monodromy representation* $\rho_D : \mathscr{B}_n \to \mathrm{Aut}(F_{n-1})$ *is well-defined and given algebraically by*

$$\rho_D(\alpha_1) : \begin{cases} v_1 \mapsto v_1 \\ v_j \mapsto v_1^{-1} v_j \quad j \neq 1, \end{cases}$$

and, for $2 \leqslant i \leqslant n-1$,

$$\rho_D(\alpha_i) : \begin{cases} v_{i-1} \mapsto v_i \\ v_i \mapsto v_i v_{i-1}^{-1} v_i \\ v_j \mapsto v_j \quad j \notin \{i-1, i\}. \end{cases}$$

(2) *We have* $A(D_n) \cong F_{n-1} \rtimes_{\rho_D} \mathscr{B}_n$, *where the projection onto the second factor is* π_D *and the section* $\mathscr{B}_n \hookrightarrow F_{n-1} \rtimes \mathscr{B}_n$ *is just* s_D. *In particular,* $\ker \pi_D$ *is a free group of rank* $n-1$.

Proof. One shows easily (by direct calculation) that this algebraic formulation of ρ_D gives a well-defined representation, and corresponds to the topological description above after identifying F_{n-1} with $\pi_1\Sigma_D$ via $v_i = [\gamma_i]$ for $i = 1, \dots, n-1$. (The latter exercise is best carried out by looking at the braid twist action on $\pi_1(\Sigma^+)$.) We turn to Part (2).

Let $\varphi : \{\delta_1, \dots, \delta_n\} \to F_{n-1} \rtimes \mathscr{B}_n$ be the function defined by $\varphi(\delta_1) = v_1\alpha_1$ and $\varphi(\delta_i) = \alpha_{i-1}$ for $i = 2, \dots, n$. One easily verifies that φ extends to a homomorphism $\varphi : A(D_n) \to F_{n-1} \rtimes \mathscr{B}_n$.

Let $\psi : \{v_1, \dots, v_{n-1}\} \cup \{\alpha_1, \dots, \alpha_{n-1}\} \to A(D_n)$ be defined by

$$\psi(v_i) = \delta_{i+1}\delta_i \dots \delta_3\delta_1\delta_2^{-1}\delta_3^{-1} \dots \delta_i^{-1}\delta_{i+1}^{-1} \quad \text{for } i = 1, \dots, n-1.$$
$$\psi(\alpha_i) = \delta_{i+1} \quad \text{for } i = 1, \dots, n-1.$$

Again, it is easily seen that ψ induces a homomorphism $\psi : F_{n-1} \rtimes \mathscr{B}_n \to A(D_n)$. Finally, one checks that $\varphi \circ \psi = \mathrm{id}$ and $\psi \circ \varphi = \mathrm{id}$, and hence that φ is an isomorphism. □

Remark. The choice of the free basis for F_{n-1} used here is the most convenient for the purposes of this section. A different free basis, given by the elements $g_1 := x_1x_2 = v_1$ and $g_i := x_ix_{i+1} = v_{i-1}^{-1}v_i$ for $i = 2, 3, \dots, n-1$, will be used later in Section 4. With respect to this basis, ρ_D is defined by

$$\rho_D(\alpha_i) : \begin{cases} g_{i-1} \mapsto g_{i-1}g_i \\ g_{i+1} \mapsto g_i^{-1}g_{i+1} \\ g_j \mapsto g_j \quad j \notin \{i-1, i+1\}. \end{cases}$$

Faithfulness of the braid monodromy representation ρ_D was established in [24] by essentially topological means (using the work of Birman and Hilden [5] on mapping class groups). We proceed now to give a purely algebraic proof of faithfulness based on the above description. We first recall some background and prove a lemma.

If T is a subset of the standard generators of an Artin group $A = A(\Gamma)$, then the subgroup A_T of A generated by T is known as a *standard parabolic subgroup* or *special subgroup* of A. By a well-known result of Van der Lek [26], each standard parabolic subgroup A_T of $A(\Gamma)$ is canonically isomorphic to the Artin group $A(\Gamma_T)$ associated to the full subgraph Γ_T of Γ spanned by T.

Lemma 2.4. *Let A be an Artin group of spherical type with standard generating set S. Let $T \subset S$, $\alpha \in S \backslash T$ and $f, g \in A_T$. If $\alpha^{-1}f\alpha = g$, then $f = g \in A_{T\cap\alpha^\perp}$, where $\alpha^\perp = \{\beta \in S : m_{\alpha,\beta} = 2\}$.*

Proof. In this proof we use the *orthogonal normal forms* for elements in a spherical type Artin group [10, 11]. Any spherical type Artin group admits a left-invariant lattice order $(A, <, \vee, \wedge)$ with positive cone the submonoid A^+ generated by the standard generators. Thus, for $x, y \in A$, $x < y$ if and only if $x^{-1}y \in A^+$. This lattice order restricts nicely to standard parabolic subgroups: $A^+ \cap A_T$ is the submonoid generated by T (written A_T^+) for any $T \subset S$. As a result of this lattice structure, every element $x \in A$ has a unique *orthogonal normal form* $x = x_1^{-1}x_2$ where $x_1, x_2 \in A^+$ and $x_1 \wedge x_2 = 1$. In fact, $x_1 = (1 \wedge x)^{-1}$ and $x_2 = (1 \wedge x)^{-1}x$. Clearly also, if $x \in A_T$ then both $x_1, x_2 \in A_T^+$.

Write $f = f_1^{-1}f_2$ and $g = g_1^{-1}g_2$ in orthogonal normal forms. We have $f_1, f_2, g_1, g_2 \in A_T^+$. The equality $\alpha^{-1}f\alpha = g$ implies that $(f_1\alpha)^{-1}(f_2\alpha) = g_1^{-1}g_2$. Put $h = f_1\alpha \wedge f_2\alpha$. Since $g_1 \wedge g_2 = 1$, we have $f_1\alpha = hg_1$ and $f_2\alpha = hg_2$. We now use the (nontrivial) fact that the submonoid A^+ of A is presented as a monoid as follows:

$$A^+ = \langle S \mid w(\alpha, \beta : m_{\alpha,\beta}) = w(\beta, \alpha : m_{\alpha,\beta}) \text{ for all } \alpha, \beta \in S\rangle^+.$$

In other words, any two positive words representing the same element of A^+ are related by a finite sequence of applications of the given relators. Using this, we see that, since $f_1 \in A_T^+$ and $\alpha \in S \backslash T$, any positive word representing $f_1\alpha$ is of the form $\gamma_1\gamma_2\dots\gamma_p.\alpha.\delta_1\delta_2\dots\delta_q$ where $\gamma_1, \dots, \gamma_p \in T$ and $\delta_1, \dots, \delta_q \in T \cap \alpha^\perp$. Now, since $f_1\alpha = hg_1$ and α does not appear in any positive word representing g_1, it follows that g_1 is written $\delta_1\delta_2\dots\delta_q$ with $\delta_1, \dots, \delta_q \in T \cap \alpha^\perp$. Therefore $g_1 \in A_{T\cap\alpha^\perp}^+$. Similarly we have $g_2 \in A_{T\cap\alpha^\perp}^+$ and therefore $f = \alpha^{-1}g\alpha = g = g_1^{-1}g_2 \in A_{T\cap\alpha^\perp}$. □

The following is the "A_n case" of Theorem 1 of [24]. A completely analogous argument can also be used to prove that Artin's representation $\rho_B : \mathscr{B}_n \to \mathrm{Aut}(F_n)$ is faithful.

Proposition 2.5 (Perron–Vannier [24]). *The representation* $\rho_D : \mathscr{B}_n \to \mathrm{Aut}(F_{n-1})$ *is faithful.*

Proof. We consider the spherical type Artin group $A = A(D_n)$. Recall the presentation $A(D_n) = F_{n-1} \rtimes_{\rho_D} \mathscr{B}_n$ where $\mathscr{B}_n$ is identified with the standard parabolic subgroup $A_{\{\delta_2,\dots,\delta_n\}}$ of A. To prove the proposition it suffices to show that, if $f \in A_{\{\delta_2,\dots,\delta_n\}}$ commutes with v_k for all $k = 1, 2, \dots, n-1$, then $f = 1$.

Let $f \in A_{\{\delta_2,\delta_3,\dots,\delta_n\}}$ such that $fv_1 = v_1 f$. Since $v_1 = \delta_1\delta_2^{-1}$, we have $\delta_1^{-1} f \delta_1 = \delta_2^{-1} f \delta_2 \in A_{\{\delta_2,\delta_3,\dots,\delta_n\}}$. By Lemma 2.4, it follows that $f \in A_{\{\delta_2,\delta_4,\dots,\delta_n\}}$.

We now show the following statement by induction on k for $k \geqslant 3$:

(I_k) If f commutes with $v_1, v_2, \dots, v_{k-1}$, then $f \in A_{\{\delta_{k+2},\delta_{k+3},\dots,\delta_n\}}$.

Suppose $k = 3$. Since f commutes with v_2 and $v_2 = \delta_3 v_1 \delta_3^{-1}$, we have that $\delta_3^{-1} f \delta_3$ commutes with v_1. From the preceding argument we deduce that both f and $\delta_3^{-1} f \delta_3 \in A_{\{\delta_2,\delta_4,\dots,\delta_n\}}$, and then, by Lemma 2.4, that $f \in A_{\{\delta_5,\delta_6,\dots,\delta_n\}}$.

Suppose $k \geqslant 4$. Since f commutes with v_{k-1} and $v_{k-1} = \delta_k v_{k-2} \delta_k^{-1}$, we have that $\delta_k^{-1} f \delta_k$ commutes with v_{k-2}. On the other hand, $\delta_k^{-1} f \delta_k$ also commutes with $v_1, v_2, \dots, v_{k-3}$ (since both δ_k and f do). By the induction hypothesis we therefore deduce that both f and $\delta_k^{-1} f \delta_k \in A_{\{\delta_{k+1},\delta_{k+2},\dots,\delta_n\}}$, and then, by Lemma 2.4, that $f \in A_{\{\delta_{k+2},\delta_{k+3},\dots,\delta_n\}}$.

The assertion (I_k) with $k = n-1$ implies that if f commutes with all $v_1, v_2, \dots, v_{n-2}$ then $f = 1$. □

2.4 Comparison of the two representations. Recall that, given a Coxeter graph Γ, the canonical map $A(\Gamma) \to W(\Gamma)$ is determined by adding the relations $\alpha^2 = 1$, for each standard generator $\alpha \in S$, to the standard presentation of $A(\Gamma)$. Via this map, the semidirect product structures $A(B_n) = F_n \rtimes \mathscr{B}_n$ and $A(D_n) = F_{n-1} \rtimes \mathscr{B}_n$ induce semidirect product structures on the associated Coxeter groups:

$$W(B_n) = C_2^n \rtimes \mathscr{S}_n \quad \text{and} \quad W(D_n) = C_2^{n-1} \rtimes \mathscr{S}_n$$

respectively, where $\mathscr{S}_n$ denotes the symmetric group on $\{1, \dots, n\}$. These product structures are, of course, well-known, see [6]. The group $W(B_n)$, sometimes called the *signed permutation group*, is simply the group of symmetries of an n-cube spanned by an orthonormal basis in $\mathbb{R}^n$: the normal subgroup C_2^n acts by change of signs of the n coordinates, and $\mathscr{S}_n$ acts by permuting the coordinates, and so acts on C_2^n by permuting the direct factors. There exists a well-known embedding $p : W(D_n) \to W(B_n)$ which commutes with the projections onto $\mathscr{S}_n$ and which sends C_2^{n-1} onto the kernel of the map $C_2^n \to C_2$ which is nontrivial on each factor. It is tempting to wonder whether the inclusion p is actually induced by an inclusion $\phi : A(D_n) \to A(B_n)$ which commutes with the projections π_B and π_D onto $\mathscr{B}_n$. This, however, is not the case.

Proposition 2.6. *There is no embedding* $\phi : A(D_n) \to A(B_n)$ *such that* $\pi_D = \pi_B \circ \phi$.

The proof of Proposition 2.6 makes use of the following lemma from Dyer and Grossman [18]. We first fix some notation: F_n is the free group of rank n generated by $u_1, u_2, \dots, u_n$ and $\mathcal{B}_n$ acts on F_n via the representation ρ_B as in Proposition 2.1—for convenience we identify $\mathcal{B}_n$ with its image under ρ_B, thus simplifying the notation.

For $x, y \in F_n$ we write $x \sim y$ if x and y are conjugate.

Lemma 2.7 (Dyer–Grossman [18], Lemma 18). *Let* $x \in F_n$ *such that* $\alpha_i^2(x) \sim x$ *for all* $i = 1, 2, \dots, n-1$. *Then x is conjugate either to a power of* u_j *for some* $j \in \{1, 2, \dots, n\}$ *or to a power of* $u_1 u_2 \dots u_n$.

Corollary 2.8. *Let* $x \in F_n$ *such that* $\alpha_i(x) \sim x$ *for all* $i = 1, \dots, n-1$. *Then x is conjugate to a power of* $u_1 u_2 \dots u_n$.

Proof. By Lemma 2.7, x is conjugate either to a power of u_j for some $j \in \{1, 2, \dots, n\}$ or to a power of $u_1 u_2 \dots u_n$. If $x \sim u_j$, then we would have $\alpha_i(u_j) \sim u_j$ for all i, while in fact $\alpha_j(u_j) = u_{j+1}$ for $j < n$ and $\alpha_{j-1}(u_j) \sim u_{j-1}$ for $j > 0$, a contradiction. □

Proof of Proposition 2.6. We suppose that such an embedding $\phi : A(D_n) \hookrightarrow A(B_n)$ exists and shall henceforth identify $A(D_n)$ with a subgroup of $A(B_n)$ via this map ϕ. Recall that, by Proposition 2.1, $A(B_n) = F_n \rtimes \mathcal{B}_n$ where $F_n = \ker \pi_B$ is the free group on generators $u_1, u_2, \dots, u_n$, $\mathcal{B}_n$ is the subgroup of $A(B_n)$ generated by $\beta_2, \beta_3, \dots, \beta_n$, and one has $\beta_i x \beta_i^{-1} = \rho_B(\alpha_{i-1})(x) = \alpha_{i-1}(x)$ for all $i = 2, 3, \dots, n$ and $x \in F_n$. Recall also (Proposition 2.3) that $A(D_n) = F_{n-1} \rtimes \mathcal{B}_n$ where $F_{n-1} = \ker \pi_D$ is the free group on generators $v_1, v_2, \dots, v_{n-1}$, $\mathcal{B}_n$ is the subgroup of $A(D_n)$ generated by $\delta_2, \delta_3, \dots, \delta_n$, and one has $\delta_i y \delta_i^{-1} = \rho_D(\alpha_{i-1})(y)$ for all $i = 2, 3, \dots, n$ and $y \in F_n$. The fact that $\pi_D = \pi_B \circ \phi$ means that $F_{n-1} \subset F_n$ and, for all $i = 2, 3, \dots, n$, there is a $w_i \in F_n$ such that $\delta_i = w_i \beta_i$.

Step 1. There exists $x_0 \in F_{n-1} \backslash \{1\}$ such that $\alpha_i(x_0) \sim x_0$ for all $i = 1, 2, \dots, n-1$.

Define
$$x_0 = \begin{cases} v_1 v_2^{-1} v_3 v_4^{-1} \dots v_{n-2}^{-1} v_{n-1} & \text{if } n \text{ even,} \\ v_1 v_2^{-1} v_3 \dots v_{n-2} v_{n-1}^{-1} v_1^{-1} v_2 v_3^{-1} \dots v_{n-2}^{-1} v_{n-1} & \text{if } n \text{ odd.} \end{cases}$$

It is easily checked that $\delta_i x_0 \delta_i^{-1} = \rho_D(\alpha_{i-1})(x_0) = x_0$. Therefore $\alpha_{i-1}(x_0) = \beta_i x_0 \beta_i^{-1} = w_i^{-1} \delta_i x_0 \delta_i^{-1} w_i = w_i^{-1} x_0 w_i$ for all $i = 2, \dots, n$.

For $x \in F_n$ we denote by $[x]$ the class of x in $H_1(F_n) \cong \mathbb{Z}^n$.

Step 2. We have $[x] = 0$ for all $x \in F_{n-1}$.

We show that $[v_i] = 0$ by induction on i. Suppose that $i = 1$. We have

$$v_1^{-2} v_2 = \rho_D(\alpha_1^2)(v_2) = \delta_2^2 v_2 \delta_2^{-2} = (w_2 \beta_2)^2 v_2 (w_2 \beta_2)^{-2} = w_2 \alpha_1(w_2) \alpha_1^2(v_2) \alpha_1(w_2^{-1}) w_2^{-1}$$

and α_1^2 acts trivially on $H_1(F_n)$, whence $[v_2] = -2[v_1] + [v_2]$, and so $[v_1] = 0$.

Suppose now that $i \geqslant 2$ and $[v_{i-1}] = 0$. Then

$$\begin{aligned}[v_i] = [\delta_i v_{i-1} \delta_i^{-1}] &= [\beta_i v_{i-1} \beta_i^{-1}] \quad \text{since } \delta_i = w_i \beta_i \text{ with } w_i \in F_n \\ &= [\alpha_{i-1}(v_{i-1})] = \alpha_{i-1}([v_{i-1}]) = 0.\end{aligned}$$

End of proof. Let x_0 be the element constructed in Step 1. By Corollary 2.8, there exists some $k \in \mathbb{Z}$ such that $x_0 \sim (u_1 u_2 \dots u_n)^k$. Step 2 now implies that

$$0 = [x_0] = k([u_1] + [u_2] + \cdots + [u_n]).$$

Since the $[u_i]$ are independent nontrivial generators of $H_1(F_n) \cong \mathbb{Z}^n$, we must have $k = 0$, and therefore $x_0 = 1$, a contradiction. $\square$

The fact that the pure braid group acts trivially on homology ($H_1(F_n) \cong \mathbb{Z}^n$) via Artin's representation figures strongly in the work of Dyer and Grossman on the automorphisms of $\mathscr{B}_n$. We note here that the action of the braid group via the braid monodromy is somewhat more complicated on the homology $H_1(F_{n-1}) \cong \mathbb{Z}^{n-1}$. In fact, this action on $\mathbb{Z}^{n-1}$ turns out to be a specialisation of the reduced Burau representation of $\mathscr{B}_n$.

3 A topological interpretation for the semidirect products

Let Γ be a Coxeter graph of spherical type (i.e. such that the Coxeter group $W(\Gamma)$ is finite). Every finite Coxeter group has a canonical representation $\theta : W(\Gamma) \hookrightarrow O(n, \mathbb{R})$ where n is the number of vertices of Γ, and where the standard generators are sent to reflections. It is well-known (since Brieskorn [7]) that each Artin group $A(\Gamma)$ of spherical type is isomorphic to the fundamental group of the space of regular orbits of the representation $\theta_{\mathbb{C}}$ of $W(\Gamma)$ as a complex unitary reflection group, obtained simply by tensoring θ with $\mathbb{C}$. This regular orbit space may be easily described as the quotient of a complex hyperplane complement as follows.

Let $\mathscr{R}$ denote the set of *reflections* in $W(\Gamma)$, that is the set of $r \in W(\Gamma)$ such that $\theta(r)$ is a reflection in a hyperplane, $H(r)$ say. (Note that every reflection is actually conjugate to a standard generator of $W(\Gamma)$). The *arrangement* of Γ is the set $\mathscr{A}(\Gamma) = \{H(r) : r \in \mathscr{R}\}$ of reflecting hyperplanes of $W(\Gamma)$. The *complexification* of $\mathscr{A}(\Gamma)$ is the set $\mathscr{A}_{\mathbb{C}}(\Gamma) = \{H_{\mathbb{C}} = H \otimes \mathbb{C} : H \in \mathscr{A}(\Gamma)\}$. Note that $H_{\mathbb{C}}(r) = H(r) \otimes \mathbb{C}$ is simply the fixed set of $\theta_{\mathbb{C}}(r)$ in $\mathbb{C}^n$, and is a hyperplane in $\mathbb{C}^n$. The *complement* of $\mathscr{A}_{\mathbb{C}}(\Gamma)$ is the manifold

$$M(\Gamma) = \mathbb{C}^n \Big\backslash \left(\bigcup_{H \in \mathscr{A}(\Gamma)} H_{\mathbb{C}} \right).$$

The group $W(\Gamma)$ acts freely on $M(\Gamma)$, and the manifold $N(\Gamma) = M(\Gamma)/W(\Gamma)$ is the regular orbit space referred to above. Brieskorn [7] showed that $\pi_1 N(\Gamma) \cong A(\Gamma)$ and

Deligne later showed that $N(\Gamma)$ is a $K(A(\Gamma),1)$-space [17].[1] We refer the reader to Brieskorn's paper [7] for an explicit description of the group isomorphism.

For types B_n, D_n and A_{n-1} we have the following (this information can be derived, for instance, from the appendices of [6], where the associated root systems are laid out).

$$\begin{aligned} M(B_n) = \{(z_1, z_2, \dots, z_n) \in \mathbb{C}^n : z_i \neq \pm z_j \text{ for } 1 \leqslant i \neq j \leqslant n, \\ \text{and } z_i \neq 0, \text{ for } 1 \leqslant i \leqslant n\}, \\ M(D_n) = \{(z_1, z_2, \dots, z_n) \in \mathbb{C}^n : z_i \neq \pm z_j, \text{ for } 1 \leqslant i \neq j \leqslant n\}. \end{aligned}$$

Let $D = \text{span}\{(1,1,\dots,1)\} \subset \mathbb{C}^n$, and, for $\mathbf{z} \in \mathbb{C}^n$, let $[\mathbf{z}]$ denote the element of $\mathbb{C}^n/D$ represented by $\mathbf{z}$. Then

$$M(A_{n-1}) = \{[z_1, z_2, \dots, z_n] \in \mathbb{C}^n/D : z_i \neq z_j, \text{ for } 1 \leqslant i \neq j \leqslant n\}.$$

The group $W(A_{n-1}) = \mathscr{S}_n$ acts on $M(A_{n-1})$ by permutation of the coordinate axes, $W(B_n) = (C_2)^n \rtimes \mathscr{S}_n$ acts on $M(B_n)$ by signed permutations of the coordinate axes (i.e. $\mathscr{S}_n$ acts by permuting the coordinates and $(C_2)^n$ acts via $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n).(z_1, z_2, \dots, z_n) = (\varepsilon_1 z_1, \varepsilon_2 z_2, \dots, \varepsilon_n z_n)$ for $\varepsilon_i = \pm 1$), and $W(D_n)$ acts on $M(D_n)$ also by signed permutations of the coordinates via the inclusion $W(D_n) < W(B_n)$ described in Subsection 2.4.

We consider $M(B_n) \subset M(D_n)$ as submanifolds of $\mathbb{C}^n$ (i.e. $M(B_n)$ is simply the hyperplane complement $M(D_n)$ with additional hyperplanes removed, namely the coordinate hyperplanes). We also identify $W(D_n)$ as a subgroup of $W(B_n)$. The canonical actions of $W(B_n)$ and $W(D_n)$ on their respective hyperplane complements are therefore simultaneously induced by the action of $W(B_n) = (C_2)^n \rtimes \mathscr{S}_n$ on $\mathbb{C}^n$ by signed permutations of the coordinates.

Let $\tilde{P}_D : M(D_n) \to M(A_{n-1})$ be the map defined by $(z_1, z_2, \dots, z_n) \mapsto [z_1^2, z_2^2, \dots, z_n^2]$, and $\tilde{P}_B$ the restriction of this map to the submanifold $M(B_n) \subset M(D_n)$. Note that these maps are equivariant with respect to the canonical projections of $W(D_n)$ and $W(B_n)$ onto $\mathscr{S}_n = W(A_{n-1})$. Thus they induce, respectively, maps $P_D : N(D_n) \to N(A_{n-1})$ and $P_B : N(B_n) \to N(A_{n-1})$.

Proposition 3.1. B)(i) *The map $P_B : N(B_n) \to N(A_{n-1})$ is a locally trivial fibration. The fibre of P_B over the orbit $\mathscr{S}_n.[\xi_1, \xi_2, \dots, \xi_n]$ is naturally homeomorphic to $\mathbb{C} \setminus \{\xi_1, \xi_2, \dots, \xi_n\}$.*

(ii) *The fibration admits a section S_B and the subsequent monodromy action of $\pi_1 N(A_{n-1})$ on the fundamental group of the fibre Σ_B over the orbit $\mathscr{S}_n.[1, 2, \dots, n]$ is precisely Artin's representation ρ_B.*

---

[1] In fact this was already known to Brieskorn [8] by more or less ad hoc means in all but 5 exceptional cases, including the cases we will consider in this article. Deligne's unified treatment was given in response to Brieskorn's Séminaire Bourbaki and has significantly influenced the subsequent study of Artin groups.

(D)(i) *The map* $P_D : N(D_n) \to N(A_{n-1})$ *is a locally trivial fibration. The fibre of* P_D *over any point is naturally homeomorphic to the twofold branched covering of* $\mathbb{C}$ *branched over the set* $\{\xi_1, \xi_2, \dots, \xi_n\}$.

(ii) *The fibration admits a section* S_D *and the subsequent monodromy action of* $\pi_1 N(A_{n-1})$ *on the fundamental group of the fibre* Σ_D *over the orbit* $\mathscr{S}_n.[1, 2, \dots, n]$ *is precisely the braid monodromy representation* ρ_D.

Proof. We first treat the B_n case. Let

$$\hat{M} = M(B_n)/(C_2)^n = \{(z_1, z_2, \dots, z_n) \in \mathbb{C}^n : z_i \neq z_j, z_i \neq 0, \text{ for } 1 \leqslant i \neq j \leqslant n\},$$

where the covering $M(B_n) \to \hat{M}$ is defined by the map $(z_1, z_2, \dots, z_n) \mapsto (z_1^2, z_2^2, \dots, z_n^2)$. Now $N(B_n) = \hat{M}/\mathscr{S}_n$ and P_B is induced by the map $\hat{P}_B : \hat{M} \to M(A_{n-1})$ given by $\mathbf{z} \mapsto [\mathbf{z}]$. Thus $\hat{P}_B$ is simply the restriction of the linear map $d : \mathbb{C}^n \to \mathbb{C}^n/D$ to the set $\hat{M}$, and is easily seen to be a locally trivial fibration where the fibre over a point $[\xi]$ is the set $(\xi + D)\backslash\{\mathbf{z} : z_i = 0 \text{ for some } i\}$, thus naturally homeomorphic to $\mathbb{C}\backslash\{\xi_1, \xi_2, \dots, \xi_n\}$ by the map sending $z \in \mathbb{C}\backslash\{\xi_1, \dots, \xi_n\}$ to the point $(\xi_1 - z, \xi_2 - z, \dots, \xi_n - z)$. Factoring out by the action of $\mathscr{S}_n$ one obtains statement (i) for P_B.

Furthermore, $\hat{P}_B$ has a section, which can be described as follows. For $\mathbf{z} = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$, define the "centre" $c_{\mathbf{z}} := \frac{1}{n}\sum z_i$ and "radius" $r_{\mathbf{z}} := \max\{|z_i - c_{\mathbf{z}}|\}$ of the set $\{z_1, z_2, \dots, z_n\}$. Then define $\hat{S}_B : M(A_{n-1}) \to \hat{M}$ such that $\hat{S}_B([\mathbf{z}]) = (z_1 + b_{\mathbf{z}}, z_2 + b_{\mathbf{z}}, \dots, z_n + b_{\mathbf{z}})$, where $b_{\mathbf{z}} := r_{\mathbf{z}} + 1 - c_{\mathbf{z}}$. Despite appearances, $\hat{S}_B$ is well-defined and continuous. Clearly, since $\hat{S}_B$ is $\mathscr{S}_n$-equivariant, it induces a section $S_B : N(A_{n-1}) \to N(B_n)$ to the map P_B. Note that the fibre over the orbit $\mathscr{S}_n.[1, 2, \dots, n] \in N(B_n)$ is naturally homeomorphic to the surface Σ_B of Figure 3, and that $\hat{S}_B([1, 2, \dots, n]) = (1, 2, \dots, n)$ and corresponds under this homeomorphism to the basepoint at 0 in $\Sigma_B = \mathbb{C}\backslash\{1, 2, \dots, n\}$.

To see that the monodromy action is precisely Artin's braid action, one simply recalls that the space of braids may be identified with the space of loops in $\{\{z_1, z_2, \dots, z_n\} \in \mathbb{C}^n/\mathscr{S}_n : z_i \neq z_j, \text{ for } 1 \leqslant i \neq j \leqslant n\}$ based at the point $\{1, 2, \dots, n\}$. This permits an identification of the braid group $\mathscr{B}_n = A(A_{n-1})$ with $\pi_1 N(A_{n-1})$ which happens to agree with that described in [7]. (Note that the standard generator α_i of $A(A_{n-1})$ is identified with the elementary braid which twists the i^{th} and $i + 1^{st}$ strands).

Now observe that the space $\hat{M}^+ = M(D_n)/(C_2)^n$ is simply obtained from $\hat{M}$ by restoring the coordinate hyperplanes. The map $M(D_n) \to \hat{M}^+$ (defined by $(z_1, z_2, \dots, z_n) \mapsto (z_1^2, z_2^2, \dots, z_n^2)$) is a ramified covering with a singular set of degree 2, the union of restored hyperplanes. Thus $\hat{M}^+$ should be thought of as an orbifold. The fibration $\hat{P}_B$ extends to an "orbifold fibration" $\hat{P}^+ : \hat{M}^+ \to M(A_{n-1})$ with the fibre over a point $[\xi]$ naturally homeomorphic to the orbifold $\mathbb{C}$ with singular set $\{\xi_1, \xi_2, \dots, \xi_n\}$ of degree 2 points (and therefore homeomorphic to the orbifold Σ^+ of Figure 5). This fibration descends to a fibration $P^+ : N^+ = \hat{M}^+/\mathscr{S}_n \to N(A_{n-1})$ with the same fibre. Just as for P_B, the fibration P^+ admits a section S^+ and the monodromy action is simply that induced by Artin's braid action on the punctured surface by replacing punctures by points of degree 2. This is precisely the braid twist action of Σ^+ described in Section 2.3.

It remains simply to observe that $N(D_n)$ is a twofold cover of N^+ (in fact $M(D_n)/\mathscr{E}_{n-1}$ is clearly a 2-fold cover of $\hat{M}^+$) which fibres over $N(A_{n-1})$ as described in the statement (i), and moreover to recall that the braid twist action on the orbifold Σ^+ lifts to the braid monodromy representation on Σ_D. The existence of the section S_D (a lift of S^+) is ensured simply by the fact that the braid twist action on Σ^+ lifts. ☐

Remark. The homotopy exact sequence implies that these fibrations give rise to semidirect product structures on the two Artin groups $\pi_1 N(B_n) \cong A(B_n)$ and $\pi_1 N(D_n) \cong A(D_n)$. Here we have not paid very much attention to explicit generators for the fundamental groups, however the careful reader may verify that the canonical isomorphisms, described in Brieskorn [7], carry the product structure coming from the fibration in each case precisely onto the semidirect product structures given in Propositions 2.1 and 2.3.

Remark. In a sense, Proposition 3.1 is just a rephrasing of some previously known facts. As explained in Allcock [1], the spaces $N(B_n)$, and N^+, may be regarded as the configuration spaces of n unordered points in the orbifolds $\mathbb{C}\backslash\{0\}$, and $\mathbb{C}$ with a degree 2 singular point, respectively. Thus $A(B_n)$ may be identified with the n-string braid group over a punctured plane (which is equally the subgroup of the $(n+1)$-string braid group in which the first string is pure), and $A(D_n)$ with an index two subgroup of the braid group over a plane with a degree 2 point. The fibrations observed above correspond simply to deleting the orbifold features (the puncture or the degree 2 point) and Artin's representation appears quite naturally from this point of view as well. We note that this braid picture for type B_n had been previously observed [21, 16] and, as Allcock pointed out, was already implicit in [8]. In fact the fibration of type D_n in Proposition 3.1 is also implicit in [8] where Brieskorn observes that the map $f : \mathbb{C}^n \to \mathbb{C}^{n-1}$ defined by $(z_1, z_2, \dots, z_n) \mapsto (z_1^2 - z_n^2, z_2^2 - z_n^2, \dots, z_{n-1}^2 z_n^2)$ restricts to a locally trivial (differentiable) fibration of $M(D_n)$. This fibration is simply the map $\tilde{P}_D$, where the image $f(M(D_n))$ is just $M(A_{n-1})$ expressed in the inhomogeneous coordinates obtained by setting the last coordinate to 0. We should point out that Allcock also gave similar "braid picture" interpretations of the infinite families of affine type Artin groups. The picture for $A(\tilde{A}_n)$ has already been used in [12], and there is certainly potential for these to be explored more fully.

4 Automorphisms preserving the fibre

This section is largely inspired by the paper of Dyer and Grossman [18] which gave the first proof, using essentially algebraic techniques, that $\mathrm{Out}(\mathscr{B}_n)$ is the group of two elements (the nontrivial outer automorphism being that which changes the sign of each standard braid group generator). While we are not yet able to replicate that result for the groups $A(B_n)$ and $A(D_n)$, we are able to prove that there are very few automorphisms of these groups which leave invariant the kernel of the map π_B and

π_D respectively (that is, the normal subgroups in the semidirect product structures described in the previous sections). The question remains as to whether these kernels are characteristic subgroups. On the other hand, we are able to show that $\mathrm{Out}((*_n C_2) \rtimes \mathscr{B}_n)$ is also of order 2.

In our investigation in this section of the automorphisms of the group $A(B_n)$ we shall always assume $n \geqslant 3$. The case $n = 2$, which was treated in [20] together with the other Artin groups of rank 2, is slightly exceptional in that the group $A(B_2)$ has a non-inner automorphism derived from the nontrivial automorphism of the graph B_2—such an automorphism does not exist if $n \geqslant 3$.

4.1 Automorphisms of $A(B_n)$ leaving F_n invariant. Recall the presentation $A(B_n) = F_n \rtimes_{\rho_B} \mathscr{B}_n$ of Section 2. For notational convenience, we shall identify elements of $\mathscr{B}_n < A(B_n)$ with the corresponding elements of $\mathrm{Inn}(A(B_n))$ via their action by conjugation (and using the fact that ρ_B is faithful). We call these elements *braid automorphisms*. As in Section 2, let F_n be freely generated by $u_1, u_2, \ldots, u_n$ so that ρ_B is defined as in Proposition 2.1. For simplicity we write $u_0 := u_1 u_2 u_3 \ldots u_n$, and note that all braid automorphisms leave u_0 fixed.

Let $\zeta \in \mathscr{B}_n$ denote the braid[2] of Figure 6. It is known that ζ generates the centre of $\mathscr{B}_n$ (it is the square of Garside's so-called fundamental element), see [19]. Also, it is easily seen that ζ acts on $A(B_n)$ by conjugation by the element u_0^{-1}, and it is a straightforward exercise to check that the centre of $A(B_n)$ is generated by the element $u_0\zeta$.

Note that there is an obvious bijective correspondence between automorphisms of $A(B_n)$ which fix the subgroup $\mathscr{B}_n$ while leaving the subgroup F_n invariant, and automorphisms of F_n which are equivariant with respect to the $\mathscr{B}_n$ action. The proof of the following result is essentially contained in the work of Dyer and Grossman [18]—see the proof of their Theorem 19.

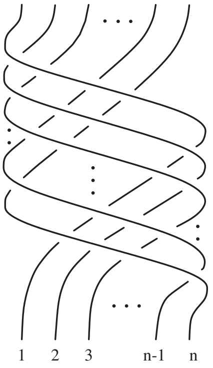

Figure 6. The braid $\zeta = (\alpha_1\alpha_2\alpha_3 \ldots \alpha_{n-1})^n$ in $\mathscr{B}_n$.

---

[2] Note that braids are drawn from top to bottom, so that their action (on $\pi_1\Sigma_B$ or $\pi_1\Sigma^+$) should be visualised by moving loops drawn in the bottom level surface up to the top level surface via a continuous path of loops.

Proposition 4.1. *Let $n \geqslant 3$, and let $\mathscr{B}_n$ act on F_n via Artin's representation ρ_B. Then the group of $\mathscr{B}_n$-equivariant automorphisms of F_n is generated by ζ.*

Proof. We first note that the group $F_n^{\mathscr{B}_n}$ of elements of F_n left fixed by $\mathscr{B}_n$ is generated by u_0: if x is a nontrivial element of $F_n^{\mathscr{B}_n}$ then, by Corollary 2.8, x is conjugate to a nontrivial power of u_0, say $x = wu_0^r w^{-1}$. But then the commutator $[w, u_0^r]$ lies in $F_n^{\mathscr{B}_n}$ and so is conjugate to a power of u_0. However, since u_0 maps to an infinite order element under abelianisation of F_n we must have $[w, u_0^r] = 1$, in which case w must be a power of u_0 and $x \in \langle u_0 \rangle$.

Let φ be a $\mathscr{B}_n$-equivariant automorphism of F_n. It follows from the above statement that φ must leave invariant the cyclic group $\langle u_0 \rangle$. Thus $\varphi(u_0) = u_0^v$ for some $v \in \{\pm 1\}$.

We now use the fact that, for all $i = 1, 2, \dots, n-1$ and all $j = 0, 1, 2, \dots, n$, we have $\alpha_i^2(u_j) \sim u_j$, and therefore by $\mathscr{B}_n$-equivariance $\alpha_i^2(\varphi(u_j)) = \varphi(\alpha_i^2(u_j)) \sim \varphi(u_j)$. It follows from Lemma 2.7, that φ must permute the $2n$ conjugacy classes of the elements $u_1^{\pm 1}, \dots, u_n^{\pm 1}$ (while respecting the pairs $\{u_i, u_i^{-1}\}$). Note that the elements $u_0^{\pm 1}, u_1^{\pm 1}, \dots, u_n^{\pm 1}$ represent distinct conjugacy classes in F_n, and none of these elements are proper powers. The conjugacy classes of u_0 and u_0^{-1} are already either fixed or interchanged.

So, for $j = 1, 2, \dots, n$, we have $\varphi(u_j) \sim u_{\sigma(j)}^{v_j}$ for $v_j \in \{\pm 1\}$, where σ denotes a permutation in $\mathscr{S}_n$. Moreover, after abelianising F_n to $\mathbb{Z}^n$, the equation $\varphi(u_0) = \varphi(u_1)\varphi(u_2)\dots\varphi(u_n)$ implies that $v_j = v$ for every $j = 1, 2, \dots, n$. It now follows, by Proposition 2.2, that there exists a braid $\beta \in \mathscr{B}_n$ such that $\beta^{-1}\varphi(u_j) = u_j^v$ for $0 \leqslant j \leqslant n$.

Now, if $v = -1$, we would have

$$u_1^{-1}u_2^{-1}\dots u_n^{-1} = \beta^{-1}\varphi(u_1u_2\dots u_n) = \beta^{-1}\varphi(u_0) = u_0^{-1} = u_n^{-1}\dots u_2^{-1}u_1^{-1},$$

which is impossible. Therefore φ agrees on F_n with the braid automorphism β. But then, since φ is $\mathscr{B}_n$-equivariant and since the monodromy representation is faithful, β must lie in the centre of $\mathscr{B}_n$ which is generated by ζ. □

Theorem 4.2. *Let $n \geqslant 3$ and consider the presentation of $A(B_n) = F_n \rtimes_{\rho_B} \mathscr{B}_n$ as a semidirect product. The group* $\mathrm{Aut}(A(B_n), F_n)$ *of automorphisms of $A(B_n)$ which leave invariant the subgroup F_n is generated by the inner automorphisms of $A(B_n)$ and the automorphism ϵ_n which simply inverts each of the standard generators. Thus*

$$\mathrm{Aut}(A(B_n), F_n) = \mathrm{Inn}(A(B_n)) \rtimes \langle \epsilon_n \rangle \cong (A(B_n)/Z(A(B_n))) \rtimes C_2,$$

where $Z(A(B_n))$ denotes the centre of $A(B_n)$. In particular, $\mathrm{Out}(A(B_n), F_n) \cong C_2$.

Proof. Let $\varphi \in \mathrm{Aut}(A(B_n), F_n)$, and let $\bar{\varphi}$ denote the automorphism induced on the quotient $\mathscr{B}_n = A(B_n)/F_n$. Using the theorem of Dyer and Grossman [18], we may suppose, up to a multiplication of φ by a braid automorphism, and by ϵ_n if necessary, that $\bar{\varphi}$ is the trivial automorphism of $\mathscr{B}_n$. In other words, there exists some function $k : \mathscr{B}_n \to F_n$, written $\gamma \mapsto k_\gamma$, such that

$$\varphi(\gamma) = k_\gamma \gamma, \quad \text{for all } \gamma \in \mathscr{B}_n.$$

In particular, for all $\gamma \in \mathscr{B}_n$ and all $x \in F_n$, we have

$$\varphi \circ \gamma(x) = \varphi(\gamma)\varphi(x)\varphi(\gamma)^{-1} = k_\gamma \gamma \varphi(x) \gamma^{-1} k_\gamma^{-1} \sim \gamma \circ \varphi(x),$$

where $\sim$ denotes conjugacy in F_n. In other words, the action of φ on the conjugacy classes of F_n is $\mathscr{B}_n$-equivariant.

Note that, for every $\gamma \in \mathscr{B}_n$, we have $\gamma(\varphi(u_0)) \sim \varphi(\gamma(u_0)) = \varphi(u_0)$. By Corollary 2.8 and the fact that u_0 is not a proper power, we then have $\varphi(u_0) \sim u_0^v$, for $v \in \{\pm 1\}$. But then, modifying φ by an inner automorphism coming from F_n, we may suppose that $\varphi(u_0) = u_0^v$. (Note that, under this modification, $\bar{\varphi}$ remains trivial, however the function k may change).

As a consequence, we now claim that φ actually fixes every element of $\mathscr{B}_n$ (and so its action on F_n is genuinely $\mathscr{B}_n$-equivariant). All braid automorphisms leave u_0 fixed. So, for any $\gamma \in \mathscr{B}_n$ we have

$$u_0^v = \varphi(u_0) = \varphi(\gamma u_0 \gamma^{-1}) = k_\gamma \gamma u_0^v \gamma^{-1} k_\gamma^{-1} = k_\gamma u_0^v k_\gamma^{-1}.$$

That is, k_γ commutes with u_0 and so must be a power of u_0, for every $\gamma \in \mathscr{B}_n$. In fact, $k : \mathscr{B}_n \to \langle u_0 \rangle \cong \mathbb{Z}$ must be a homomorphism since $\varphi(\gamma\beta) = k_\gamma \gamma k_\beta \beta = k_\gamma k_\beta \gamma \beta$, using once again the fact that braid automorphisms fix powers of u_0. Moreover, in view of the braid relations $\alpha_i \alpha_{i+1} \alpha_i = \alpha_{i+1} \alpha_i \alpha_{i+1}$, the only homomorphisms $\mathscr{B}_n \to \mathbb{Z}$ are multiples of the length homomorphism $\ell : \mathscr{B}_n \to \mathbb{Z}$ which sends each standard generator to 1. In other words, there exists an $m \in \mathbb{Z}$ such that $\varphi(\gamma) = u_0^{m\ell(\gamma)}\gamma$, for all $\gamma \in \mathscr{B}_n$. Finally, we observe that the centre of $A(B_n)$, which is generated by the element $u_0\zeta$, is left invariant by any automorphism of $A(B_n)$. So $\varphi(u_0\zeta) = (u_0\zeta)^{\pm 1}$. But since $\varphi(u_0\zeta) = u_0^v u_0^{m\ell(\zeta)}\zeta$ we must have $m\ell(\zeta) = 1 - v$. But, since $\ell(\zeta) = n(n-1) \geqslant 6$, this is only possible if $m = 0$, in which case φ fixes every element of $\mathscr{B}_n$ as claimed.

It now follows from Proposition 4.1 that φ is a power of the central braid automorphism ζ, and hence an inner automorphism of $A(B_n)$. □

From the proof we see that there exist sections of the map $\pi_B : A(B_n) \to \mathscr{B}_n$ which are "exotic" in the the sense that they are distinct, up to automorphism of $A(B_n)$, from the standard section. Namely, for each $m \neq 0$, setting $\gamma \mapsto u_0^{m\ell(\gamma)}\gamma$ defines such a section. We do not know if there exist exotic sections other than these. A similar remark also applies to the fibration of $A(D_n)$ considered in the next section (see the proof of Theorem 4.9).

4.2 Automorphisms of $A(D_n)$ leaving F_{n-1} invariant. We proceed now to derive the analogous result for the type D_n Artin group. The proof of Theorem 4.2 is largely transportable to this case. Thus, the main part of our work will be in establishing the following analogue of Proposition 4.1.

Theorem 4.3. *Let $n \geqslant 4$, and identify $\mathcal{B}_n$ with a subgroup of* $\mathrm{Aut}(F_{n-1})$ *via the braid monodromy representation ρ_D. Then the group of $\mathcal{B}_n$-equivariant automorphisms of F_{n-1} is generated by ζ.*

Let $K = \langle x_1, x_2, \ldots, x_n \mid x_i^2 = 1, i = 1, 2, \ldots, n\rangle$ denote the group $\pi_1\Sigma^+ = *_n(C_2)$, where the generators x_i are as described in Subsection 2.3. The braid group $\mathcal{B}_n$ acts on K (via braid twists of the degree 2 orbifold points) in such a way that, for $i = 1, 2, \ldots, n-1$,

$$\alpha_i : \begin{cases} x_i \mapsto x_{i+1} \\ x_{i+1} \mapsto x_{i+1}x_i x_{i+1} \\ x_j \mapsto x_j \quad \text{if } j \notin \{i, i+1\} \end{cases}$$

The braid monodromy action (via ρ_D) is just the restriction of this action to the characteristic index 2 free subgroup $F_{n-1} < K$, namely the kernel of the map of K onto C_2 which maps each generator x_i nontrivially. In this section we shall use the following set of free generators for F_{n-1}: $\{g_j = x_j x_{j+1} : j = 1, 2, \ldots, n-1\}$.

We first study the action of $\mathcal{B}_n$ on the whole of K. For $u, v \in K$ we write $u \sim v$ to mean that u and v are conjugate in K, and write $[u]$ for the conjugacy class of u in K. Note that any element in K is uniquely represented by a *reduced form*, a word in $x_1, \ldots, x_n$ with all exponents $+1$. In addition, every conjugacy class in K has a representative whose reduced form is *cyclically reduced*, i.e. every cyclic shift of the word is also reduced (equivalently, the word is reduced and does not begin and end with the same letter). Such a representative is unique up to cyclic shifts of its reduced form, and shall be called a *shortest* element of its conjugacy class. The same observations hold in the case of reduced forms with respect to free products of arbitrary groups. The following three lemmas shall be also used in the next subsection, so we shall assume $n \geqslant 2$ in their statement in place of $n \geqslant 4$.

Lemma 4.4. *Let $1 \leqslant j \leqslant n-1$, and let $K^{\langle\alpha_j\rangle}$ denote the subgroup of K consisting of those elements left fixed by α_j.*

(i) *$K^{\langle\alpha_j\rangle}$ is generated by the elements $\{g_j = x_j x_{j+1}, x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_n\}$.*

(ii) *If $w \in K$ is such that $\alpha_j(w) \sim w$, then there is a shortest element of $[w]$ lying in $K^{\langle\alpha_j\rangle}$.*

Proof. (i) The group K may be written $C * D$ where $C = \langle x_j, x_{j+1}\rangle$ and $D = \langle x_i : i \neq j, j+1\rangle$. Clearly α_j fixes every element of D and leaves C invariant. So $K^{\langle\alpha_j\rangle} = C^{\langle\alpha_j\rangle} * D$.

For simplicity we write $\alpha = \alpha_j$, $x = x_j$ and $y = x_{j+1}$. We have $\alpha(x) = y$ and $\alpha(y) = yxy$, thus $\alpha(xy) = xy$ and $\alpha(yx) = yx$. Any element of $C = \langle x\rangle * \langle y\rangle \cong C_2 * C_2$ may be written in normal form with respect to the free product and has one of the following forms: $(xy)^k, (yx)^k = (xy)^{-k}, (xy)^k x$, or $y(xy)^k$, for some $k \in \mathbb{N}$. Now $\alpha((xy)^k x) = (xy)^k y \neq (xy)^k x$, and $\alpha(y(xy)^k) = yxy(xy)^k \neq y(xy)^k$. Thus $C^{\langle\alpha\rangle}$ is generated by $xy = g_j$, and $K^{\langle\alpha_j\rangle} = \langle g_j\rangle * D$ is as claimed.

(ii) We may clearly assume $w \neq 1$ and may choose w to be any shortest element of its conjugacy class. Thus, either $w \in C$, or $w \in D$, or w may be chosen to have reduced form $W = c_1 d_1 c_2 d_2 \dots c_m d_m$ with respect to the free product $C * D$, where the c_i are nontrivial elements of C and the d_i are nontrivial elements of D, and $m \geqslant 1$.

Suppose first that $w \in C$. Since it is shortest in its conjugacy class, w must be represented by a cyclically reduced word in the letters x, y. That is, w is either $(xy)^k$ or $(yx)^k = (xy)^{-k}$, for some $k \in \mathbb{N}$. But then w lies in $K^{\langle \alpha_j \rangle}$.

If $w \in D$ then, a fortiori, w is an element of $K^{\langle \alpha_j \rangle}$.

Finally we suppose that w has reduced form $W = c_1 d_1 c_2 d_2 \dots c_m d_m$ (with respect to $C * D$), in which case it follows that $W' := \alpha(c_1) d_1 \dots \alpha(c_m) d_m$ is the reduced form for $\alpha(w)$ with respect to $C * D$. Since $\alpha(w) \sim w$, and since both W and W' are cyclically reduced, we must have that W' is obtained from W by a cyclic shift. That is, there exists an $r \in \{1, 2, \dots, m\}$ such that $\alpha(c_i) = c_{i+r}$ and $d_i = d_{i+r}$ for all $i = 1, 2, \dots, m$ (where indices are taken mod m). But then $\alpha^{2m}(c_i) = c_i$ for all $i = 1, 2, \dots, m$. Note that α^2 acts on both x and y by conjugating by the element yx. But then $\alpha^{2m}(c_i) = c_i$ only if c_i is a power of xy. It follows that w must lie in $K^{\langle \alpha_j \rangle}$. □

The following lemma should be compared with Lemma 18 of [18] (see Lemma 2.7) and, more particularly, with Corollary 2.8.

Lemma 4.5. (i) *The group $K^{\mathscr{B}_n}$ of elements of K left fixed by every $\gamma \in \mathscr{B}_n$ is the cyclic group generated by the element $\delta := x_1 x_2 \dots x_n$.*

(ii) *If $w \in K$ is such that $\alpha_j(w) \sim w$ for all $j = 1, 2, \dots, n-1$, then w is conjugate to an element of $K^{\mathscr{B}_n}$ (so conjugate to a power of δ).*

Proof. The case $n = 2$ is contained in Lemma 4.4. We assume $n \geqslant 3$.

(i) Let $w \in K^{\mathscr{B}_n}$ and let $u_1 u_2 \dots u_k$ be the reduced form for w (each $u_i \in \{x_1, x_2, \dots, x_n\}$).

(1) Suppose that $u_i = x_j$ where $1 < j < n$. Then, by Lemma 4.4, the inclusions $w \in K^{\langle \alpha_{j-1} \rangle}$ and $w \in K^{\langle \alpha_j \rangle}$ imply that $1 < i < k$ and $u_{i-1} u_i u_{i+1} = x_{j-1} x_j x_{j+1}$, or $x_{j+1} x_j x_{j-1}$.

(2) Suppose that $u_i = x_1$. Then, by Lemma 4.4, the inclusion $w \in K^{\langle \alpha_1 \rangle}$ implies that either $u_{i-1} = x_2$ or $u_{i+1} = x_2$. Moreover, we cannot have both identities, $u_{i-1} = x_2$ and $u_{i+1} = x_2$, otherwise, by (1), $x_3 x_2 x_1 x_2 x_3$ would be a subexpression of $u_1 u_2 \dots u_k$ (recall that $n \geqslant 3$), and then the inclusion $w \in K^{\langle \alpha_1 \rangle}$ would contradict Lemma 4.4.

(3) Suppose $u_i = x_n$. Then, by Lemma 4.4, the inclusion $w \in K^{\langle \alpha_{n-1} \rangle}$ implies that either $u_{i-1} = x_{n-1}$ or $u_{i+1} = x_{n-1}$. Moreover, we cannot have both identities, $u_{i-1} = x_{n-1}$ and $u_{i+1} = x_{n-1}$, otherwise, by (1), $x_{n-2} x_{n-1} x_n x_{n-1} x_{n-2}$ would be a subexpression of $u_1 u_2 \dots u_k$, and then the inclusion $w \in K^{\langle \alpha_{n-1} \rangle}$ would contradict Lemma 4.4.

Clearly, (1), (2), and (3) imply that w is a power of δ. On the other hand, it is easily verified that δ is indeed fixed by every braid.

(ii) Let $w \in K$ such that $\alpha_j(w) \sim w$ for all $j = 1, \dots, n-1$, and let $u_1 u_2 \dots u_k$ be a cyclically reduced word representing an element of $[w]$ (each $u_i \in \{x_1, x_2, \dots, x_n\}$).

(1) Suppose $u_i = x_j$, where $1 < j < n$. Then, by Lemma 4.4, we have $u_{i-1}u_iu_{i+1} = x_{j-1}x_jx_{j+1}$, or $x_{j+1}x_jx_{j-1}$. (This time, the indices of the u_i's are considered modulo k. So $u_0 = u_k$ and $u_{k+1} = u_1$.)

(2) Suppose $u_i = x_1$. Then, by Lemma 4.4, either $u_{i-1} = x_2$ or $u_{i+1} = x_2$, and we cannot have both identities, $u_{i-1} = x_2$ and $u_{i+1} = x_2$.

(3) Suppose $u_i = x_n$. Then, by Lemma 4.4, either $u_{i-1} = x_{n-1}$ or $u_{i+1} = x_{n-1}$, and we cannot have both identities, $u_{i-1} = x_{n-1}$ and $u_{i+1} = x_{n-1}$.

Now, (1), (2), and (3) clearly imply that $u_1u_2\dots u_k$ is a power of δ up to a cyclic shift. □

We will only need to use Part (i) of the following lemma, but we state Part (ii) anyway for the sake of completeness.

Lemma 4.6. *Let S denote the set of standard generators of $\mathcal{B}_n$. Choose a sequence $1 \leqslant i_1 < i_2 < \cdots < i_k < n$ and let $T \subset S$ be the set $T = S \backslash \{\alpha_j : j = i_1, i_2, \dots, i_k\}$. Let $K^{\langle T \rangle}$ denote the subgroup of K consisting of those elements left fixed by every element of T. Then*:

(i) $K^{\langle T \rangle} = \langle \delta(1, i_1), \delta(i_1 + 1, i_2), \delta(i_2 + 1, i_3), \dots, \delta(i_k + 1, n) \rangle$, *where, for* $i < j$, $\delta(i, j) := x_ix_{i+1}\dots x_{j-1}x_j$.

(ii) *If $w \in K$ is such that $\alpha_j(w) \sim w$ for all $\alpha_j \in T$, then w is conjugate to an element of $K^{\langle T \rangle}$.*

Proof. (i) Write $i_0 = 1$ and $i_{k+1} = n$. Consider the decomposition $K = K_0 * K_1 * \cdots * K_k$, where K_r is the subgroup of K generated by $x_{i_k+1}, \dots, x_{i_{k+1}}$. Let $w \in K^{\langle T \rangle}$, and let $w = w_1w_2\dots w_p$ be the reduced form of w with respect to this decomposition, namely, $w_i \in K_{\mu(i)}$ for some $\mu(i) \in \{0, 1, \dots, k\}$, and $\mu(i+1) \neq \mu(i)$ for all $i = 1, \dots, p-1$. Now, by exactly the argument of Lemma 4.5 (i), we deduce that, if $w_i \in K_r$ (namely, $r = \mu(i)$), then w_i is a power of $\delta(i_r + 1, i_{r+1})$.

(ii) This follows by a similar extension of Lemma 4.5 (ii). □

We will also make use of the following two lemmas in the proof of Theorem 4.3.

Lemma 4.7. *Let φ be an automorphism of the free group $F\langle x, y \rangle$ such that $\varphi(y) = y^v$ for some $v \in \{\pm 1\}$. Then $\varphi(x) = y^kx^\varepsilon y^l$ for some $k, l \in \mathbb{Z}$ and $\varepsilon \in \{\pm 1\}$.*

Proof. Write $\varphi(x) = y^kuy^l$ where $k, l \in \mathbb{Z}$ and u is a nontrivial reduced word whose first and last letters are either x or x^{-1}. We will use the fact that, since φ is an automorphism, the group $F\langle x, y \rangle$ is generated by y and u. The reduced word u is written uniquely in the form $u_0u_1u_0^{-1}$ where the subword u_1 is cyclically reduced and nontrivial. It follows that, for any $r \in \mathbb{Z} \backslash \{0\}$, u^r has reduced form $u_0u_1^ru_0^{-1}$. Thus the reduced form for u^r begins and ends in x or x^{-1} and has length at least length(u).

Let $w(s, t)$ be a reduced word in the letters s, t involving at least one t. The above observation shows that the length of $w(y, u)$ is greater than or equal to the length of u. So, x can never lie in the subgroup generated by y and u unless u is of length 1. But then $\varphi(x)$ must be of the form $\varphi(x) = y^kx^\varepsilon y^l$ with $\varepsilon \in \{\pm 1\}$. □

Lemma 4.8. *Let $w(s,t)$ denote a freely reduced word in the letters s, t. Then, in the free group $F\langle x,y\rangle$, we have the relation $w(x,xy).w(y,xy)=1$ if and only if $w(s,t)$ is the trivial word.*

Proof. Clearly the relation holds if $w(s,t)$ is the trivial word. Suppose then that the relation holds, and let $u(s,t)$ be the freely reduced word in s, t such that $u(x,y)=w(x,xy)$. Then we have

$$u(x,y)y^{-1}u(y,x)y = w(x,xy)y^{-1}w(y,yx)y = w(x,xy)w(y,xy) = 1.$$

But then $y^{-1}u(y,x)y = u(x,y)^{-1}$. Since $u(y,x)$ and $u(x,y)^{-1}$ are reduced words of the same length, then either they are both trivial, or the word $u(x,y)^{-1}$ is obtained from the word $u(y,x)$ by a cyclic shift involving only the letter y. But the latter is impossible since if $u(y,x)$ starts (respectively ends) with $y^{\pm1}$, then $u(x,y)^{-1}$ ends (respectively starts) with $x^{\pm1}$. So $w(x,xy)=u(x,y)=1$. But since mapping $s\mapsto x$ and $t\mapsto xy$ defines an isomorphism $F\langle s,t\rangle\to F\langle x,y\rangle$, $w(x,xy)=1$ only if $w(s,t)$ is the trivial word. □

Proof of Theorem 4.3. The proof of Theorem 4.3 falls into two cases, depending on whether n is even or odd. Observe that the braid ζ acts on F_{n-1} by conjugation by the element δ^{-1} where $\delta := x_1x_2\dots x_n\in K$, and that $\delta\in F_{n-1}$ if and only if n is even.

Proof of Theorem 4.3, case *n* even. Let $\varphi: F_{n-1}\to F_{n-1}$ be some $\mathcal{B}_n$-equivariant automorphism. Our objective is to prove that φ is a power of ζ. The first observation is that φ (*and* its inverse) must leave invariant the fixed subgroup $F_{n-1}^{\Gamma}=K^{\Gamma}\cap F_{n-1}$ for any $\Gamma<\mathcal{B}_n$. That is, φ restricts to an automorphism of F_{n-1}^{Γ}.

By Lemma 4.5, $F_{n-1}^{\mathcal{B}_n}=K^{\mathcal{B}_n}=\langle\delta\rangle$. Therefore, by the above observation, φ restricts to an automorphism of $\langle\delta\rangle$, hence $\varphi(\delta)=\delta$ or δ^{-1}. We also have, from Lemma 4.6, that the two elements g_1 and δ freely generate the subgroup $F_{n-1}^{\langle T\rangle}=K^{\langle T\rangle}$, where $T=\{\alpha_1\}\cup\{\alpha_3,\dots,\alpha_{n-1}\}$. Thus φ restricts to an automorphism $\varphi: F\langle g_1,\delta\rangle\to F\langle g_1,\delta\rangle$. Lemma 4.7 now applies to show that $\varphi(g_1)=\delta^k g_1^{\varepsilon}\delta^l$ for some $k,l\in\mathbb{Z}$ and $\varepsilon\in\{\pm1\}$.

We now observe that, for each $i=2,3,\dots,n-1$, $g_i=\gamma_i(g_1)$ for some braid $\gamma_i\in\mathcal{B}_n$. Thus by equivariance of φ and the fact that δ is fixed by all braids we have

$$\varphi(g_i)=\delta^k g_i^{\varepsilon}\delta^l \quad \text{for fixed } k,l\in\mathbb{Z} \text{ and } \varepsilon=\pm1, \text{ and for all } i.$$

Now, since n is even, we have $\delta=g_1g_3\dots g_{n-1}\in F_{n-1}$. Therefore,

$$\varphi(\delta)=\delta^k g_1^{\varepsilon}\delta^{l+k}g_3^{\varepsilon}\delta^{l+k}\dots\delta^{l+k}g_{n-1}^{\varepsilon}\delta^l=\delta^{\nu} \quad \text{for some } \nu=\pm1. \tag{1}$$

Abelianising F_{n-1} to $\mathbb{Z}^{n-1}$ the relation (1) becomes $(m(l+k)+\varepsilon-\nu)[\delta]=0$, where $n=2m\geqslant4$. There are two cases:

(i) $\varepsilon=\nu$ and therefore $l+k=0$, or

(ii) $\varepsilon=-\nu$ and therefore $m.(l+k)=2\nu$, so that, necessarily, $n=4$ and $l+k=\nu$.

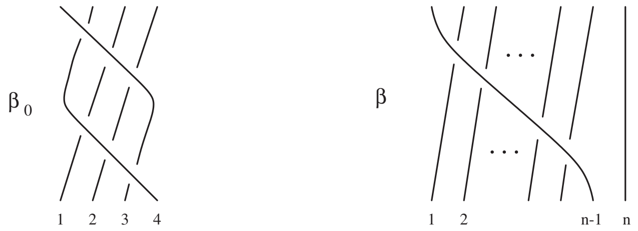

Figure 7. The braids $\beta_0 = (\alpha_1\alpha_2\alpha_3)^2$ and $\beta = \alpha_1\alpha_2\dots\alpha_{n-2}$.

In Case (i), the possibility $\varepsilon = \nu = -1$ leads to a contradiction, for it implies that $g_1g_3\dots g_{n-1} = g_{n-1}\dots g_3g_1$. But then $\varepsilon = 1$, and φ evidently just acts on F_{n-1} by conjugation by δ^k. That is, $\varphi = \zeta^{-k}$ as required.

In Case (ii), $n = 4$, $\delta = g_1g_3$, $l + k = \nu = -\varepsilon$, and (1) simplifies to $g_1^\varepsilon(g_1g_3)^{-\varepsilon}g_3^\varepsilon = 1$, which is possible if and only if $\varepsilon = -1$. But then, $\zeta^k\varphi$ is the automorphism $F_3 \to F_3$ sending g_i to $g_i^{-1}\delta$ for $i = 1, 2, 3$. That is $g_1 \mapsto g_3$, $g_2 \mapsto g_2^{-1}g_1g_3$ and $g_3 \mapsto g_3^{-1}g_1g_3$. Let $\beta_0 \in \mathcal{B}_4$ denote the braid $(\alpha_1\alpha_2\alpha_3)^2$, shown in Figure 7. Then one can easily check that $\zeta^k\varphi$ is realised by the action of β_0, which is a contradiction since β_0 does not lie in the centre of $\mathcal{B}_4$ and so is not $\mathcal{B}_4$-equivariant (while $\zeta^k\varphi$ is).

Proof of Theorem 4.3, case *n* odd. As in the previous case, let $\varphi : F_{n-1} \to F_{n-1}$ be some $\mathcal{B}_n$-equivariant automorphism. Again, we shall prove that φ is a power of ζ, using the observation that φ restricts to an automorphism of $F_{n-1}^\Gamma = K^\Gamma \cap F_{n-1}$ for any $\Gamma < \mathcal{B}_n$.

Define $x := x_1x_2x_3\dots x_{n-1}$. Then $x = g_1g_3\dots g_{n-2} \in F_{n-1}$ and $\delta = xx_n$. Note that this time δ does not lie in F_{n-1}, however the element $z := \delta^2 = xx_nxx_n$ does. Therefore $F_{n-1}^{\mathcal{B}_n} = \langle z\rangle$ (since, by Lemma 4.5, $K^{\mathcal{B}_n} = \langle\delta\rangle$). Since φ restricts to an automorphism of this group, we have $\varphi(z) = z$ or z^{-1}.

Claim 1. *After multiplication of φ by a power of ζ we may suppose that $\varphi(x) = x$.*

Proof. We consider the subset of braid generators $T = \{\alpha_1, \alpha_2, \dots, \alpha_{n-2}\}$. By Lemma 4.6, $K^{\langle T\rangle} = \langle x\rangle * \langle x_n\rangle \cong \mathbb{Z} * C_2$, and $F_{n-1}^{\langle T\rangle}$ is therefore freely generated by the two elements x and $\hat{x} := x_nxx_n$. Note that $x\hat{x} = xx_nxx_n = \delta^2 = z$, so that $F_{n-1}^{\langle T\rangle}$ is also freely generated by x and z. Thus φ restricts to an automorphism of $\langle x, z\rangle \cong F_2$. Since $\varphi(z) = z$ or z^{-1}, Lemma 4.7 applies to give

$$\varphi : \begin{cases} x \mapsto z^k x^\varepsilon z^l \\ z \mapsto z^\nu \end{cases}$$

for some $k, l \in \mathbb{Z}$ and $\varepsilon, \nu \in \{\pm 1\}$.

Notice that $\zeta(x) = \delta^{-1}x\delta = \hat{x}$ (since $\delta = xx_n$). Consequently, by equivariance, we have that $\varphi(\hat{x}) = z^k\hat{x}^\varepsilon z^l$. Thus

$$\varphi(z) = \varphi(x\hat{x}) = z^k x^\varepsilon z^{l+k} \hat{x}^\varepsilon z^l = z^v. \tag{2}$$

Abelianising F_2 to $\mathbb{Z}^2$, this gives $(2(k+l) + \varepsilon - v)[z] = 0$. There are two cases:

(i) $\varepsilon = v$ and $l + k = 0$, or

(ii) $\varepsilon = -v$ and $(l + k) = -\varepsilon$.

In Case (i) above, (2) becomes $x^\varepsilon \hat{x}^\varepsilon = (x\hat{x})^\varepsilon$ which is only possible if $\varepsilon = 1$ in which case $\varphi(x) = z^k x z^{-k} = \zeta^{-2k}(x)$.

In Case (ii) above, (2) becomes $x^\varepsilon (x\hat{x})^{-\varepsilon} \hat{x}^\varepsilon = 1$ which is only possible if $\varepsilon = -1$, and so $l + k = 1$. In this case $\varphi(x) = z^k x^{-1} z^l = \zeta^{-2k}(x^{-1}z) = \zeta^{-2k}(\hat{x}) = \zeta^{-2k+1}(x)$, since $\zeta(x) = \hat{x}$. □

We now restrict our attention to the subgroup $H = \langle g_1, g_2, g_3, \ldots, g_{n-2} \rangle * \langle x_n \rangle \cong F_{n-2} * C_2$ of K. Let $\hat{\ }: H \to H$ denote the involution which is conjugation by x_n. (This is consistent with the definition of the element $\hat{x}$ already introduced). Then we see that $H \cap F_{n-1} = A * B$ where A denotes the group $\langle g_1, g_2, \ldots, g_{n-2} \rangle \cong F_{n-2}$ and $B := \hat{A}$. Thus B is freely generated by the elements $\hat{g}_1, \hat{g}_2, \hat{g}_3, \ldots, \hat{g}_{n-2}$. (Note: $A * B$ is just the kernel of the map $H \to C_2$ induced by the quotient $K \to K/F_{n-1}$). Given a free product of groups $G = G_1 * G_2$, each element $g \in G$ has a unique expression in *reduced form* with respect to the free product decomposition, namely an expression $g = u_1 u_2 u_3 \ldots u_k$ where each syllable u_i is a nontrivial element of either G_1 or G_2 and where two consecutive syllables do not belong to the same factor G_1 or G_2. The number k shall be referred to as the *syllable length* of g and written $\|g\|$.

Consider the set of braid generators $T = \{\alpha_1\} \cup \{\alpha_3, \ldots, \alpha_{n-2}\}$. By Lemma 4.6

$$K^{\langle T \rangle} = \langle g_1, g_3 g_5 \ldots g_{n-2} \rangle * \langle x_n \rangle = \langle g_1, x \rangle * \langle x_n \rangle \cong F_2 * C_2.$$

Let $A_1 := \langle g_1, x \rangle$ and $B_1 := \hat{A}_1 = \langle \hat{g}_1, \hat{x} \rangle$. Then $K^{\langle T \rangle} \cap F_{n-1} = A_1 * B_1$, and φ restricts to an automorphism of this group. In particular, $\varphi(g_1) \in A_1 * B_1$. Let

$$W = w_1 w_2 w_3 \ldots w_k$$

be the reduced form for $\varphi(g_1)$ with respect to $A_1 * B_1$. Thus the syllables w_i are nontrivial elements coming alternately from the groups A_1 and B_1. Since $A_1 < A$ and $B_1 < B$, this is equally a reduced form with respect to $A * B$. Also W is nontrivial since $\varphi(g_1)$ is nontrivial.

Now let $\beta \in \mathcal{B}_n$ denote the braid $\alpha_1 \alpha_2 \ldots \alpha_{n-2}$ shown in Figure 7. Observe that $\beta(g_i) = g_{i+1}$ if $i \leqslant n-3$ and $\beta(x) = x$ (so that $\beta(g_{n-2}) = (g_2 g_4 \ldots g_{n-3})^{-1} g_1 g_3 \ldots g_{n-2}$ using the fact that $x = g_1 g_3 \ldots g_{n-2}$). Thus β leaves the subgroup A invariant. Moreover, since $\beta(x_n) = x_n$, the braid β leaves the whole of $A * B$ invariant respecting the free product structure. Thus, for instance, $\beta^r(W) = \beta^r(w_1)\beta^r(w_2)\ldots\beta^r(w_k)$ is a reduced form with respect to $A * B$ for the element $\beta^r(\varphi(g_1)) = \varphi(g_{r+1})$, for all $r \in \mathbb{Z}$. Moreover, these reduced forms all have similar structure: a syllable $\beta^r(w_i)$ comes from the factor A if and only if w_i does.

Claim 2. *If* $\varphi(x) = x$, *then* $\varphi(g_1) \in A_1$.

Proof. We continue with the notation introduced in the preceding paragraphs. By the hypothesis $\varphi(x) = x$, we may use the fact that $\varphi(x)$ has syllable length 1 with respect to $A * B$. However, since $g_j = \beta^{j-1}(g_1)$, for $j = 1, 2, \dots, n-2$ (this is NOT true for $j = n-1$), and since φ acts $\mathscr{B}_n$-equivariantly, we have

$$\varphi(x) = \varphi(g_1 g_3 g_5 \dots g_{n-2}) = W\beta^2(W)\beta^4(W)\dots\beta^{n-3}(W).$$

If $\|W\|$ is even then $W\beta^2(W)\beta^4(W)\dots\beta^{n-3}(W)$ is already in reduced form with respect to $A * B$, and has syllable length at least 4 ($n \geqslant 5$ and $\|W\| \geqslant 2$) which contradicts the uniqueness of reduced forms in a free product.

We may assume therefore that $\|W\|$ is odd, and let $m = \frac{1}{2}(\|W\| + 1)$. Then W is of the form $U^{-1}w_m V$ where U and V are reduced forms of the same syllable length with respect to $A * B$ and whose first syllables are of the same *type* (i.e. they come from the same factor A or B).

Write $V = v_1 v_2 \dots v_{m-1}$ and $U = u_1 u_2 \dots u_{m-1}$ as reduced forms. If $V\beta^2(U)^{-1} \neq 1$ then there is a last $i \in \{1, 2, \dots, m-1\}$ such that $v_i \neq \beta^2(u_i)$, in which case $V\beta^2(U)^{-1}$ has reduced form

$$M = v_1 v_2 \dots v_{i-1}.\omega.\beta^2(u_{i-1}^{-1})\dots\beta^2(u_2^{-1})\beta^2(u_1^{-1})$$

where $\omega = v_i\beta^2(u_i)^{-1}$ is a nontrivial element of the same free factor as v_i. This implies that

$$U^{-1}w_m M\beta^2(w_m)\beta^2(M)\dots\beta^{n-5}(M)\beta^{n-3}(w_m)\beta^{n-3}(V)$$

is a reduced form for $\varphi(x)$ of syllable length at least 3 ($n \geqslant 5$). But this again contradicts uniqueness of the reduced form. Thus we may suppose that $V = \beta^2(U)$.

Both V and U represent elements of $A_1 * B_1$. Thus $V = \beta^2(U)$ represents an element of $(A_1 * B_1) \cap \beta^2(A_1 * B_1)$ which may also be written $\langle g_1, \hat{g}_1, x, \hat{x}\rangle \cap \langle g_3, \hat{g}_3, x, \hat{x}\rangle$. Except in the case $n = 5$ where $x = g_1 g_3$, the elements g_1, g_3, x, $\hat{g}_1$, $\hat{g}_3$, $\hat{x}$ form a free system. Therefore, in the case $n \geqslant 7$, the intersection of these two rank 4 free groups is just $\langle x, \hat{x}\rangle \cong F_2$ and is therefore fixed elementwise by β. Since V lies in this intersection, we now have $U = \beta^{-2}(V) = V$. Thus $\varphi(x)$ has reduced form $U^{-1}\omega U$ with middle syllable $\omega := w_m\beta^2(w_m)\dots\beta^{n-3}(w_m)$. Note that $\omega \neq 1$ since it is conjugate to $\varphi(x) \neq 1$. But since $\|\varphi(x)\| = 1$ we must have $U = 1$. Therefore $m = 1$ and $\varphi(g_1) = Uw_m U^{-1} = w_1$ and clearly lies in A_1 since otherwise we would have $\omega \in B$ contradicting the fact that $\omega = \varphi(x) = x \in A$. This completes the proof of the Claim in the case $n \geqslant 7$.

In the case $n = 5$, we have $\varphi(x) = U^{-1}w_m\beta^2(w_m)\beta^2(V)$ (since $V = \beta^2(U)$). Since it is a syllable, w_m is a nontrivial element of either A_1 or B_1. Suppose that $w_m \in A_1 \backslash \{1\}$. Then, there is a nontrivial freely reduced word $w(s, t)$ in the letters s, t, such that $w_m\beta^2(w_m) = w(g_1, x)w(g_3, x)$, where in this case $x = g_1 g_3$. But then, by Lemma 4.8, $\omega := w_m\beta^2(w_m) \neq 1$ and $U^{-1}\omega\beta^2(V)$ is therefore a reduced form for $\varphi(x)$. The fact that $\|\varphi(x)\| = 1$ then tells us that $U = V = 1$, so that $\varphi(g_1) = w_m \in A_1$, as required. If, however, $w_m \in B_1$, then the same argument shows that $\varphi(g_1) = w_m \in B_1$ and $\varphi(x) = w_m\beta^2(w_m) \in B$, which would contradict $\varphi(x) = x$. □

By Claim 1, we may suppose that $\varphi(x) = x$. In that case, by Claim 2, $\varphi(A_1) < A_1$. But, since $\varphi^{-1}(x) = x$, applying Claim 2 to φ^{-1} also gives $\varphi^{-1}(A_1) < A_1$, and so φ restricts to an automorphism of A_1. Lemma 4.7 now applies to give $\varphi(g_1) = x^k g_1^{\varepsilon} x^l$ for some $k, l \in \mathbb{Z}$ and $\varepsilon = \pm 1$. By equivariance of φ with respect to β, we in fact have $\varphi(g_j) = x^k g_j^{\varepsilon} x^l$ for all $j = 1, 2, \dots, n-2$ (but *not* necessarily for $j = n-1$). But then

$$x = \varphi(x) = x^k g_1^{\varepsilon} x^{l+k} g_3^{\varepsilon} x^{l+k} \dots x^{l+k} g_{n-2}^{\varepsilon} x^l$$

Abelianising F_{n-1} to $\mathbb{Z}^{n-1}$ yields the equation $\frac{(n-1)}{2}(l+k) + \varepsilon = 1$. We either have

(i) $\varepsilon = 1$ and $l + k = 0$, in which case $\varphi(g_i) = x^k g_i x^{-k}$ for $i \leqslant n-2$, or

(ii) $n = 5$, $l + k = 1$ and $\varepsilon = -1$, in which case $\varphi(g_i) = x^k g_i^{-1} x^{1-k}$ for $i = 1, 2, 3$.

Now let $y = g_2 g_4 \dots g_{n-1}$. By a similar analysis (labelling the x_i in the reverse order) we arrive at the conclusion that there exist $m, l \in \mathbb{Z}$ such that either

(iii) $\zeta^m \varphi(g_i) = y^l g_i y^{-l}$ for all $i = 2, 3, \dots, n-1$, or

(iv) $\zeta^m \varphi(g_i) = y^l g_i^{-1} y^{1-l}$ for all $i = 2, 3, 4$ (and $n = 5$).

Consider the possibilities (i)–(iv) for g_2. Since no two of g_2, $g_2^{-1}x$ and $g_2^{-1}y$ can be conjugate in K we must have (i) and (iii). Then $g_2 = (x^{-k}\delta^{-m}y^l)g_2(y^{-l}\delta^m x^k)$. The element $y^{-l}\delta^m x^k$ of K commutes with g_2, thus it must be a power of g_2. That is to say $\delta^m = y^l g_2^q x^{-k}$ for some $q \in \mathbb{Z}$. In particular, since $\delta^2, x, y, g_1 \in F_{n-1}$, but $\delta \notin F_{n-1}$, we must have m even. Now, take $x, g_2, g_3, \dots, g_{n-2}, y$ as a free basis for F_{n-1}. With respect to these generators, $z = \delta^2$ has a reduced form

$$z = xy^{-1} g_2 g_4 \dots g_{n-3} (g_2 g_3 \dots g_{n-1} g_{n-2})^{-1} g_3 g_5 \dots g_{n-2} x^{-1} y.$$

Therefore, $\delta^m = z^{m/2}$ cannot have the form $y^l g_2^q x^{-k}$ unless $k = l = q = m = 0$. So, φ must be the identity on F_{n-1}. This completes the proof of Theorem 4.3.

We shall now work with the presentation $A(D_n) = F_{n-1} \rtimes_{\rho_D} \mathcal{B}_n$ of Section 2. Note that $A(D_n)$ is an index 2 normal subgroup of $K \rtimes_{\rho^+} \mathcal{B}_n \cong \pi_1 N^+$, where $\pi_1 N^+$ denotes the orbifold fundamental group of the orbifold N^+ introduced in Section 3. (In fact $K \rtimes \mathcal{B}_n \cong A(D_n) \rtimes C_2$ where the section sends the generator of C_2 to x_1). For notational convenience, we shall identify elements of $\mathcal{B}_n < A(D_n)$ with the corresponding elements of $\mathrm{Inn}(A(D_n))$ via their action by conjugation (and using the fact that ρ_D is faithful). We call these elements *braid automorphisms*. As with the B_n case, it is easily seen that ζ acts on $K \rtimes \mathcal{B}_n$, and so on $A(D_n)$, by conjugation by δ^{-1}, where $\delta = x_1 x_2 \dots x_n$. It is also a straightforward exercise to check that the centre of $K \rtimes \mathcal{B}_n$ is generated by the element $\delta\zeta$, and that the centre of $A(D_n)$ is generated by $\delta\zeta$ if n is even, and $\delta^2\zeta^2$ if n is odd.

Special automorphisms. We define the automorphism $\epsilon_n \in \mathrm{Aut}(K \rtimes \mathscr{B}_n)$ such that $\epsilon_n(\alpha_i) = \alpha_i^{-1}$ for $i = 1, 2, \dots, n-1$, and $\epsilon_n(x_1) = x_1$. It is a short exercise to show that ϵ_n is well-defined. Moreover, ϵ_n induces the automorphism of $A(D_n)$ which, by abuse of notation, we shall also call ϵ_n, and which is defined by $\epsilon_n(\delta_i) = \delta_i^{-1}$ for every $i = 1, 2, \dots, n$.

We also define the so-called *graph automorphism* $\tau_n \in \mathrm{Aut}(A(D_n))$ induced by the involution of the Coxeter graph of type D_n, namely $\tau_n(\delta_1) = \delta_2$, $\tau_n(\delta_2) = \delta_1$, and $\tau_n(\delta_i) = \delta_i$ if $i \geqslant 3$.

Note that, since $A(D_n)$ is a normal subgroup of $K \rtimes \mathscr{B}_n$, every inner automorphism of $K \rtimes \mathscr{B}_n$ induces an automorphism of $A(D_n)$. These constitute a subgroup of $\mathrm{Aut}(A(D_n))$ which is generated by the inner automorphisms of $A(D_n)$ and conjugation by x_1 in the larger group. The latter automorphism of $A(D_n)$ is precisely the graph automorphism τ_n.

Finally, we note that the automorphism τ_n is an inner automorphism of $A(D_n)$ if and only if n is odd. This is a straightforward consequence of [23]. Alternatively, one can observe that if τ_n is an inner automorphism, i.e. conjugation by some element $k\beta$, say, where $k \in F_{n-1}$ and $\beta \in \mathscr{B}_n$, then the element $x_1 k\beta \in K \rtimes \mathscr{B}_n$ centralizes $A(D_n)$. But that is to say that conjugation by $x_1 k$ in K agrees with the action of β^{-1}, and so fixes δ. But then $x_1 k \in K \backslash F_{n-1}$ is a power of δ, which is only possible if n is odd. On the other hand, if n is odd, x_1 differs from δ by an element of F_{n-1}, and conjugation by δ is realised by the braid automorphism ζ, so τ_n is an inner automorphism of $A(D_n)$.

Theorem 4.9. *Let $n \geqslant 4$. The group $\mathrm{Aut}(A(D_n), F_{n-1})$ of automorphisms of $A(D_n)$ leaving invariant the subgroup F_{n-1} is generated by the inner automorphisms, the graph automorphism τ_n, and the automorphism ϵ_n. More precisely,*

$$\mathrm{Aut}(A(D_n), F_{n-1}) = \begin{cases} \mathrm{Inn}(A(D_n)) \rtimes \langle \epsilon_n, \tau_n \rangle \cong (A(D_n)/Z(A(D_n))) \rtimes (C_2 \times C_2) \\ \hfill \text{if } n \text{ is even} \\ \mathrm{Inn}(A(D_n)) \rtimes \langle \epsilon_n \rangle \cong (A(D_n)/Z(A(D_n))) \rtimes C_2 \\ \hfill \text{if } n \text{ is odd} \end{cases}$$

where $Z(A(D_n))$ denotes the centre of $A(D_n)$. In particular, $\mathrm{Out}(A(D_n), F_{n-1}) \cong C_2 \times C_2$ *if n is even, and* $\mathrm{Out}(A(D_n), F_{n-1}) \cong C_2$ *if n is odd.*

Proof. Let φ be an arbitrary automorphism of $A(D_n) = F_{n-1} \rtimes \mathscr{B}_n$ such that $\varphi(F_{n-1}) = F_{n-1}$. Then φ induces an automorphism $\bar{\varphi}$ of the quotient group $\mathscr{B}_n = A(D_n)/F_{n-1}$. Using the theorem of Dyer and Grossman [18], we may suppose, up to a multiplication of φ by a braid automorphism, and by ϵ_n if necessary, that $\bar{\varphi}$ is the trivial automorphism of $\mathscr{B}_n$. In other words, there exists some function $k : \mathscr{B}_n \to F_{n-1}$, written $\gamma \mapsto k_\gamma$, such that

$$\varphi(\gamma) = k_\gamma \gamma, \quad \text{for all } \gamma \in \mathscr{B}_n.$$

We now consider the element $z \in K$ which generates the subgroup of fixed points in F_{n-1} under the braid action. That is $z = \delta$, if n is even, and δ^2 if n is odd. For any $\gamma \in \mathscr{B}_n$ we have

$$\varphi(z) = \varphi(\gamma z \gamma^{-1}) = k_\gamma \gamma \varphi(z) \gamma^{-1} k_\gamma^{-1}. \tag{3}$$

That is to say, $\gamma(\varphi(z)) \sim \varphi(z)$ for all $\gamma \in \mathscr{B}_n$, where $\sim$ denotes conjugacy in K. But then Lemma 4.5, plus the fact that z is not a proper power in F_{n-1}, implies that $\varphi(z)$ is conjugate in K to z^ν, for $\nu \in \{\pm 1\}$.

Now, multiplying φ by an inner automorphism from F_{n-1} and by τ_n if necessary (that is by some automorphism which is induced by conjugation, in the bigger group $K \rtimes \mathscr{B}_n$, by an element of K), we may suppose, in fact, that $\varphi(z) = z^\nu$. Note that this modification of φ does not change the induced automorphism $\bar{\varphi}$, but it does modify the function k. In any case, the Equation (3) together with the fact that braid automorphisms fix z, gives simply $z^\nu = k_\gamma z^\nu k_\gamma^{-1}$. But then k_γ must be a power of z for every $\gamma \in \mathscr{B}_n$. In fact, $k : \mathscr{B}_n \to \langle z \rangle \cong \mathbb{Z}$ must be a homomorphism, since $\varphi(\gamma\beta) = k_\gamma \gamma k_\beta \beta = k_\gamma k_\beta \gamma \beta$. Moreover, in view of the braid relations $\alpha_i \alpha_{i+1} \alpha_i = \alpha_{i+1} \alpha_i \alpha_{i+1}$, the only homomorphisms $\mathscr{B}_n \to \mathbb{Z}$ are multiples of the length homomorphism $\ell : \mathscr{B}_n \to \mathbb{Z}$ which sends each standard generator to 1. In other words, there exists an $m \in \mathbb{Z}$ such that $\varphi(\gamma) = z^{m\ell(\gamma)}\gamma$, for all $\gamma \in \mathscr{B}_n$.

Finally, we observe that the centre of $A(D_n)$ is generated by the element $z\zeta$ if n is even, and $z\zeta^2$ if n is odd, and is left invariant by any automorphism of $A(D_n)$. For instance, if $n \geqslant 5$ is odd, $\varphi(z\zeta) = (z\zeta)^{\pm 1}$. Since $\varphi(z\zeta) = z^\nu z^{m\ell(\zeta)}\zeta$ we deduce that $m\ell(\zeta) = 1 - \nu$. But, since $\ell(\zeta) = n(n-1) \geqslant 20$, we must have $m = 0$. Similarly, when n is even we deduce that $m = 0$, and in both cases the action of φ on F_{n-1} is therefore $\mathscr{B}_n$-equivariant. By Theorem 4.3, we now have that φ agrees on F_{n-1} with a central braid automorphism ζ^k for some $k \in \mathbb{Z}$. But then, $\zeta^{-k}\varphi$ fixes both subgroups F_{n-1} and $\mathscr{B}_n$, so is trivial, and φ is an inner automorphism. □

4.3 Automorphisms of $K \rtimes \mathscr{B}_n$. We continue with the same notation as in the previous subsection. Recall that $K = *_n(C_2)$ denotes the free product of n copies of C_2, and $\mathscr{B}_n$ acts on K via the representation ρ^+ define in Section 2. We proceed now to determine the automorphism group of $K \rtimes_{\rho^+} \mathscr{B}_n$ following the same strategy as in the previous subsections. First, we establish the following result which is analogous to Proposition 4.1 for the group $A(B_n)$ and to Theorem 4.3 for the group $A(D_n)$.

Proposition 4.10. *Let $n \geqslant 3$, and identify $\mathscr{B}_n$ with a subgroup of* Aut(K) *via the representation ρ^+. Then the group of $\mathscr{B}_n$-equivariant automorphisms of K is the cyclic subgroup of* Aut(K) *generated by ζ.*

Proof. Let $\varphi : K \to K$ be a $\mathscr{B}_n$-equivariant automorphism. By Lemma 4.5, $K^{\mathscr{B}_n}$ is the infinite cyclic subgroup generated by $\delta = x_1 x_2 \dots x_n$, thus $\varphi(\delta) = \delta^\nu$ where $\nu \in \{\pm 1\}$. Let $T = \{\alpha_2, \alpha_3, \dots, \alpha_{n-1}\}$. By Lemma 4.6, we have $K^{\langle T \rangle} = \langle x_1, x_2 x_3 \dots x_n \rangle = \langle x_1 \rangle * \langle \delta \rangle \cong C_2 * \mathbb{Z}$, thus φ restricts to an automorphism $\langle x_1 \rangle * \langle \delta \rangle \to \langle x_1 \rangle * \langle \delta \rangle$. The element $\varphi(x_1)$ is of order 2 and $\varphi(x_1) \in K^{\langle T \rangle}$, thus $\varphi(x_1)$ is conjugate to x_1 in

$K^{\langle T\rangle}$, namely, there exists $w \in K^{\langle T\rangle}$ such that $\varphi(x_1) = wx_1w^{-1}$. Now, $\varphi(\delta) = \delta^{\pm 1}$ and $\varphi(x_1) = wx_1w^{-1}$ generate $K^{\langle T\rangle} = \langle x_1\rangle * \langle\delta\rangle$, and this is possible only if w is of the form $w = \delta^k x_1^\mu$ with $k \in \mathbb{Z}$ and $\mu \in \{0,1\}$, thus $\varphi(x_1) = \delta^k x_1 \delta^{-k}$ for some $k \in \mathbb{Z}$. Observe that, for $i = 1, 2, \dots, n$, there exists a braid $\gamma_i \in \mathscr{B}_n$ such that $\gamma_i(x_1) = x_i$. By equivariance, it follows that $\varphi(x_i) = \delta^k x_i \delta^{-k}$ for all $i = 1, \dots, n$, and therefore $\varphi = \zeta^{-k}$. □

Recall that ϵ_n denotes the automorphism of $K \rtimes \mathscr{B}_n$ determined by $\epsilon_n(x_1) = x_1$ and $\epsilon_n(\alpha_i) = \alpha_i^{-1}$ for $i = 1, \dots, n-1$.

Theorem 4.11. *Let $n \geqslant 2$. The group $\mathrm{Aut}(K \rtimes \mathscr{B}_n)$ is generated by the inner automorphisms and the automorphism ϵ_n. Thus*

$$\mathrm{Aut}(K \rtimes \mathscr{B}_n) = \mathrm{Inn}(K \rtimes \mathscr{B}_n) \rtimes \langle \epsilon_n \rangle \cong ((K \rtimes \mathscr{B}_n)/Z(K \rtimes \mathscr{B}_n)) \rtimes C_2,$$

where $Z(K \rtimes \mathscr{B}_n)$ denotes the centre of $K \rtimes \mathscr{B}_n$. In particular, $\mathrm{Out}(K \rtimes \mathscr{B}_n) \cong C_2$.

Proof. The case $n = 2$ is special and should be treated separately. We leave this case to the reader, and assume from now on that $n \geqslant 3$.

Let $\varphi : K \rtimes \mathscr{B}_n \to K \rtimes \mathscr{B}_n$ be an automorphism. Note that K is the smallest subgroup of $K \rtimes \mathscr{B}_n$ which contains every element of order 2, and therefore is a characteristic subgroup. Let $\bar\varphi$ be the automorphism of $\mathscr{B}_n$ induced by φ. Using the theorem of Dyer and Grossman [18], we may suppose, up to a multiplication of φ by a braid automorphism and by ϵ_n if necessary, that $\bar\varphi$ is the trivial automorphism of $\mathscr{B}_n$. So, there exists a function $k : \mathscr{B}_n \to K$, $\gamma \mapsto k_\gamma$, such that $\varphi(\gamma) = k_\gamma\gamma$ for all $\gamma \in \mathscr{B}_n$.

Recall that every braid fixes $\delta = x_1x_2\dots x_n$. For any $\gamma \in \mathscr{B}_n$, we have

$$\varphi(\delta) = \varphi(\gamma\delta\gamma^{-1}) = k_\gamma\gamma\varphi(\delta)\gamma^{-1}k_\gamma^{-1}. \tag{4}$$

This shows that $\gamma(\varphi(\delta)) \sim \varphi(\delta)$ for all $\gamma \in \mathscr{B}_n$, thus, by Lemma 4.5, $\varphi(\delta)$ is conjugate to δ^ν for some $\nu \in \{\pm 1\}$. Multiplying φ by an inner automorphism from K, we may suppose that $\varphi(\delta) = \delta^\nu$. This modification does not change the induced automorphism $\bar\varphi$, but it does modify the function $k : \mathscr{B}_n \to K$.

The Equation (4) together with the fact that braid automorphisms fix δ, gives $\delta^\nu = k_\gamma\delta^\nu k_\gamma^{-1}$, thus k_γ is a power of δ. It follows that the function $k : \mathscr{B}_n \to \langle\delta\rangle \cong \mathbb{Z}$ is a homomorphism since, for $\gamma, \beta \in \mathscr{B}_n$, k_β being a power of δ, we have $\varphi(\gamma\beta) = k_\gamma\gamma k_\beta\beta = k_\gamma k_\beta\gamma\beta$. The only homomorphisms $\mathscr{B}_n \to \mathbb{Z}$ are multiples of the length function $\ell : \mathscr{B}_n \to \mathbb{Z}$, thus there exists $m \in \mathbb{Z}$ such that $k_\gamma = \delta^{m\ell(\gamma)}$ for all $\gamma \in \mathscr{B}_n$.

Recall that the centre of $K \rtimes \mathscr{B}_n$ is the infinite cyclic subgroup generated by $\delta\zeta$. In particular, we have $\varphi(\delta\zeta) = (\delta\zeta)^{\pm 1}$. Since $\varphi(\delta\zeta) = \delta^\nu\delta^{m\ell(\zeta)}\zeta$, it follows that $m\ell(\zeta) = 1 - \nu$, thus $m = 0$ (since $\ell(\zeta) \geqslant 6$). This shows that $\varphi(\gamma) = \gamma$ for all $\gamma \in \mathscr{B}_n$, and therefore that the action of φ on K is $\mathscr{B}_n$-equivariant. By Proposition 4.10, φ agrees on K with a central braid automorphism ζ^k for some $k \in \mathbb{Z}$. But then, $\zeta^{-k}\varphi$ fixes both subgroups K and $\mathscr{B}_n$, so it is trivial, and φ is an inner automorphism. □

References

[1] D. Allcock, Braid pictures for Artin groups. *Trans. Amer. Math. Soc.* **354** (2002), 3455–3474. MR1911508 (2003f:20053) Zbl 01761311

[2] E. Artin, Theorie der Zöpfe. *Hamburg Abh.* **4** (1925), 47–72. JFM 51.0450.01

[3] E. Artin, Braids and permutations. *Ann. of Math.* (2) **48** (1947), 643–649. MR0020989 (9,6c) Zbl 0030.17802

[4] E. Artin, Theory of braids. *Ann. of Math.* (2) **48** (1947), 101–126. MR0019087 (8,367a) Zbl 0030.17703

[5] J. S. Birman, H. M. Hilden, On isotopies of homeomorphisms of Riemann surfaces. *Ann. of Math.* (2) **97** (1973), 424–439. MR0325959 (48 #4305) Zbl 0237.57001

[6] N. Bourbaki, *Groupes et algèbres de Lie, Chapitres IV–VI*. Hermann, Paris 1968. MR0240238 (39 #1590) Zbl 0983.17001

[7] E. Brieskorn, Die Fundamentalgruppe des Raumes der regulären Orbits einer endlichen komplexen Spiegelungsgruppe. *Invent. Math.* **12** (1971), 57–61. MR0293615 (45 #2692) Zbl 0204.56502

[8] E. Brieskorn, Sur les groupes de tresses [d'après V. I. Arnol'd]. In: *Séminaire Bourbaki, 24ème année* (1971/1972), *Exp. No.* 401, 21–44. Lecture Notes in Math., Vol. 317, Springer 1973. MR0422674 (54 #10660) Zbl 0277.55003

[9] E. Brieskorn, K. Saito, Artin-Gruppen und Coxeter-Gruppen. *Invent. Math.* **17** (1972), 245–271. MR0323910 (48 #2263) Zbl 0243.20037

[10] R. Charney, Artin groups of finite type are biautomatic. *Math. Ann.* **292** (1992), 671–683. MR1157320 (93c:20067) Zbl 0736.57001

[11] R. Charney, Geodesic automation and growth functions for Artin groups of finite type. *Math. Ann.* **301** (1995), 307–324. MR1314589 (95k:20055) Zbl 0813.20042

[12] R. Charney, D. Peifer, The $K(\pi, 1)$-conjecture for the affine braid groups. *Comment. Math. Helv.* **78** (2003), 584–600. MR1998395 (2004f:20067) Zbl 02021086

[13] A. M. Cohen, L. Paris, On a theorem of Artin. *J. Group Theory* **6** (2003), 421–441. MR2007739 (2004m:20072) Zbl 1039.20016

[14] H. S. M. Coxeter, Discrete groups generated by reflections. *Ann. of Math.* (2) **35** (1934), 588–621. MR1503182 JFM 60.0898.02

[15] H. S. M. Coxeter, The complete enumeration of finite groups of the form $R_i^2 = (R_iR_j)^{k_{ij}} = 1$. *J. London Math. Soc.* **10** (1935), 21–25. JFM 61.0097.05

[16] J. Crisp, Injective maps between Artin groups. In: *Geometric group theory down under* (*Canberra*, 1996), 119–137, de Gruyter 1999. MR1714842 (2001b:20064) Zbl 1001.20034

[17] P. Deligne, Les immeubles des groupes de tresses généralisés. *Invent. Math.* **17** (1972), 273–302. MR0422673 (54 #10659) Zbl 0238.20034

[18] J. L. Dyer, E. K. Grossman, The automorphism groups of the braid groups. *Amer. J. Math.* **103** (1981), 1151–1169. MR636956 (82m:20041) Zbl 0476.20026

[19] F. A. Garside, The braid group and other groups. *Quart. J. Math. Oxford Ser.* (2) **20** (1969), 235–254. MR0248801 (40 #2051) Zbl 0194.03303

[20] N. D. Gilbert, J. Howie, V. Metaftsis, E. Raptis, Tree actions of automorphism groups. *J. Group Theory* **3** (2000), 213–223. MR1753479 (2001a:20039) Zbl 0979.20027

[21] S. S. F. Lambropoulou, Solid torus links and Hecke algebras of B-type. In: *Proceedings of the Conference on Quantum Topology* (*Manhattan, KS*, 1993), 225–245, World Sci. Publishing, River Edge, NJ 1994. MR1309934 (96a:57020) Zbl 0884.57004

[22] W. Magnus, Über Automorphismen von Fundamentalgruppen berandeter Flächen. *Math. Ann.* **109** (1934), 617–646. JFM 60.0091.01

[23] L. Paris, Centralizers of parabolic subgroups of Artin groups of type A_l, B_l, and D_l. *J. Algebra* **196** (1997), 400–435. MR1475117 (98k:20062) Zbl 0937.20018

[24] B. Perron, J. P. Vannier, Groupe de monodromie géométrique des singularités simples. *Math. Ann.* **306** (1996), 231–245. MR1411346 (97j:32031) Zbl 0863.32013

[25] J. Tits, Normalisateurs de tores. I. Groupes de Coxeter étendus. *J. Algebra* **4** (1966), 96–116. MR0206117 (34 #5942) Zbl 0145.24703

[26] H. van der Lek, The homotopy type of complex hyperplane complements. Ph.D. thesis, Nijmegen 1983.

Received 1 October, 2003; revised 4 August, 2004

J. Crisp, L. Paris, Institut de Mathématiques de Bourgogne, Université de Bourgogne, UMR 5584 du CNRS, B.P. 47870, 21078 Dijon cedex, France
Email: {jcrisp,lparis}@u-bourgogne.fr