On hyperbolic iterated distortions for the adjustment of survival functions

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Ast&Risk 2011, Dijon, 17-18 janvier 2011

Outline

The problem

What we are looking for :
a parametric representation $\tilde{S}$ of some target survival function.
(e.g. conditional mortality, asset returns, losses...)

Classical parametric representations may have some pitfalls :

- *Improving data adequation, parameter adjunction* is not always simple (e.g. Heligman, Pollard, Lee, Carter, Wang)
- *Estimation problems* may occur (e.g. Heligman, Pollard). How to get good initial values ?
- *Analytical representation of the inverse* survival function is sometimes required.

Proposed methodology

Probability distortions

Probability distortions : long history, large context (see D'Alembert, 1768, Yaari, Eeckoudt, Wang, Hardy)

Some usages of probability distortions :

- *Improving a fit* by distorting a reference function (adjusting an official mortality table to business data, adjusting claims distribution on a segment given a global distribution).
- *Explaining a phenomenon* by the considered distortion (shape of the distortion, evolution over time of the distortion, incidence of a quantitative factor).
- *Applying a prudential rule* (loading preserving bracket pricing, solvability margin, risk measures).

Distortions we are looking for

The class of considered survival function is $\mathcal{S}$.

Distorted survival functions

Distorted functions $\tilde{S} \in \mathcal{S}$ will be built from $S \in \mathcal{S}$:

$$\forall x \in \mathbb{R}, \quad \tilde{\mathbf{S}}(\mathbf{x}) = \mathbf{T}_\theta(\mathbf{S}(\mathbf{x})) .$$

With a distortion $T_\theta : [0,1] \to [0,1]$ and $\theta \in \Theta$ a vector of parameters.

How to choose the class $\mathcal{T}$ of considered distortions ?

$$\mathcal{T} = \{ T_\theta : [0,1] \to [0,1] \}_{\theta \in \Theta} .$$

We would like "*good properties*" for :

- each $T_i \in \mathcal{T}$,
- each composite $T_1 \circ T_2 \circ \cdots \circ T_n$.

required properties (i)

Consider a distortion $T_\theta \in \mathcal{T}$,

- *C1. Invertibility*
  $(T_\theta)^{-1}$ should exist, with an explicit analytical form.

- *C2. Stability*
  $(T_\theta)^{-1}$ should belong to $\mathcal{T}$.

- *C3. Regularity*
  Partial derivatives of T_θ should be continuous :

$$\forall x \in [0,1],\ \theta \mapsto T_\theta(x) \text{ continuously differentiable },$$
$$\forall \theta \in \Theta,\ x \mapsto T_\theta(x) \text{ continuously differentiable }.$$

required properties (ii)

- *C4. Convergence*
  One should find a sequence of distortions to converge any target.

$$\forall S_0, S_1 \in \mathcal{S},\ \exists \text{ a serie } (T_i)_{i\in\mathbb{N}},\ T_n \circ \cdots \circ T_1(S_0) \longrightarrow S_1 \,.$$

- *C5. Parameterization*
  One should write $(T_\theta)^{-1} = T_{\theta'}$, with θ' easily deduced from θ :

$$\theta' = D_T \cdot \theta\,,\ \ (\textit{symmetrical parameterization})$$
$$D_T \text{ diagonal matrix, with elements in } \{-1, 1\}\ ,$$

$$\text{or } \theta' = -\theta\,,\ \ (\textit{entirely symmetrical parameterization})$$

Proposed distortions : Logit scale

Distortion from a conversion function f

For f any bijective increasing function from $\mathbb{R}$ to $\mathbb{R}$.

$$\begin{aligned} T_f &: [0,1] \to [0,1] \\ T_f(u) &= \begin{cases} 0 & \text{if } u = 0, \\ \operatorname{logit}^{-1}(f(\operatorname{logit}(u))) & \text{if } 0 < u < 1, \\ 1 & \text{if } u = 1. \end{cases} \end{aligned}$$

f is said to be the *conversion function*

The distorted function $\tilde{S}$ of a survival function S will be :

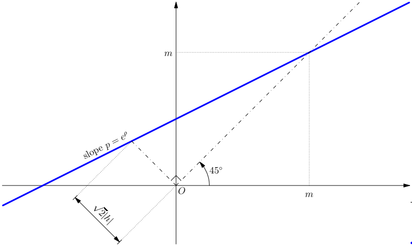
$$\tilde{S}(x) = T_f(S(x)).$$

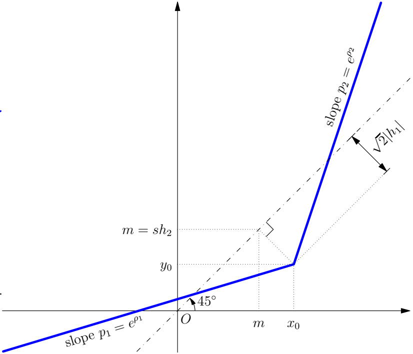
Note that :

$$T_f \circ T_g = T_{f \circ g}$$

Proposed distortions (i)

Affine and angle conversion functions f :

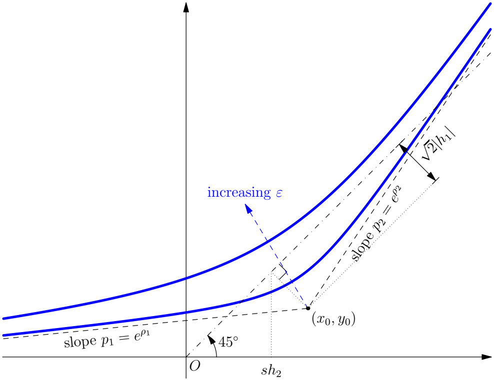




On $]0,1[$, the corresponding distortion is $T_f(u) = \mathrm{logit}^{-1}(f(\mathrm{logit}(u)))$.

Proposed distortions (ii)

Hyperbolic function : smooth version of angle functions f :



On $]0, 1[$, the corresponding distortion is $T_f(u) = \text{logit}^{-1}(f(\text{logit}(u)))$.

example of some hyperbolic distortions

A 2 parameters hyperbola (+1 smoothing parameter η)

$$
\begin{aligned}
H_{m,\rho,\eta}(x) &= m + (1+e^{\rho})\frac{x-m}{2} - (1-e^{\rho})\sqrt{\left(\frac{x-m}{2}\right)^2 + e^{\eta-\frac{\rho}{2}}} \\
(H_{m,\rho,\eta})^{-1}(x) &= H_{m,-\rho,\eta}(x)
\end{aligned}
$$

Induced 2+1 parameter distortion (with common parameter η) :

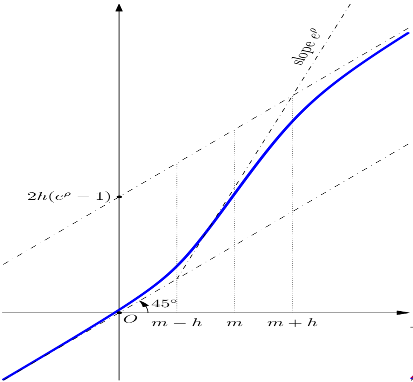
$$
\begin{aligned}
T_H(u) &= \mathrm{logit}^{-1} \circ H \circ \mathrm{logit}(u) \\
(T_H)^{-1}(u) &= \mathrm{logit}^{-1} \circ H^{-1} \circ \mathrm{logit}(u)
\end{aligned}
$$

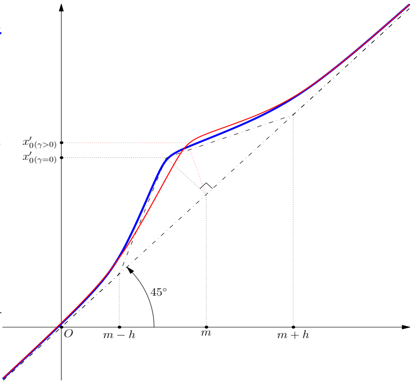
Induced 2n+1 parameters distortion (with common parameter η) :

$$
\begin{aligned}
T_G(u) &= \mathrm{logit}^{-1} \circ H_1 \circ \cdots \circ H_n \circ \mathrm{logit}(u) \\
(T_G)^{-1}(u) &= \mathrm{logit}^{-1} \circ H_n^{-1} \circ \cdots \circ H_1^{-1} \circ \mathrm{logit}(u)
\end{aligned}
$$

Proposed distortions (iii)

Composite Hyperbolic functions : smooth version of angle compositions.
Some basic composites of two hyperbolic distortions $T = T_H \circ T_H$:





On $]0,1[$, the corresponding distortion is $T_f(u) = \mathrm{logit}^{-1}(f(\mathrm{logit}(u)))$.

Results

One can show simple results for distorted function :

- Impact on r.v., on hazard rates
- Results for regular variation distorted functions
- Risk measure conditions
- Results for eliminating useless parameters of composite distortions

main results for hyperbolic composite distortion

- One can find $(T_i)_{i=1,2...}$ so that $\tilde{S}$ **converge to any target**.
- One can **estimate parameters** of any T_i, with simple and very **good initialization values**.
- One can write the **analytic expression of the inverse** $(\tilde{S})^{-1}$.

Some hints for the results

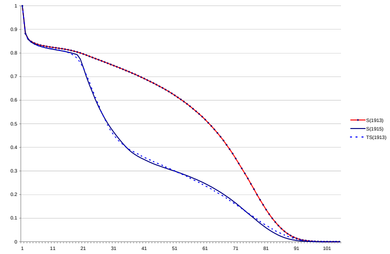
1. A survival function can be approximated by a piecewise linear function.
2. A piecewise linear function is an angle composition.
3. An angle composition is a particular hyperbolic composite distortion.

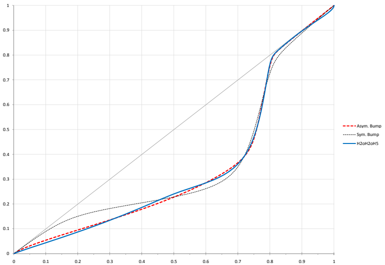
Numerical illustrations

Some Results (i)

Catastrophic event modeling :

$$S^{1915}(x) = T_f\left(S^{1913}(x)\right) .$$



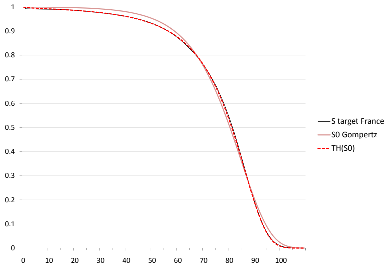


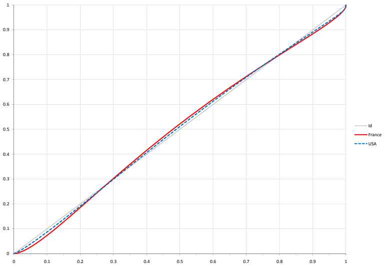
(5 parameters distortion)

Some Results (ii)

Static distortion of prospective mortality

$$S(x,t) = T_{f_\theta}(S^{Gompertz}_{a_0+\delta_a t, b_0+\delta_b t}(x)), \quad \theta = constant\ .$$



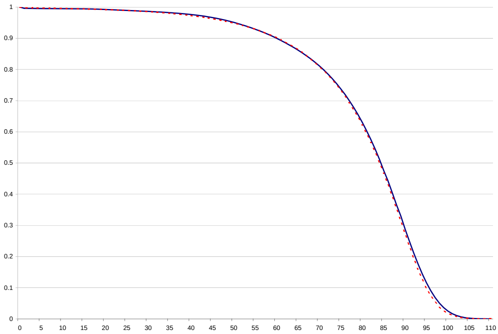


adjustment : years 1975-2005, picture : year 1990 (HMD Database)
(5 parameters distortion)

Some Results (iii)

Dynamic distortion of a static exponential law

$$S(x,t) = T_{f_{\theta_t}}(S^{expo}(x)), \quad \theta_t = \theta_0 + t\Delta_\theta\,.$$



adjustment : years 1975-2005, picture : worst adjustment year 2004
14 parameters distortion, $>$ 3000 data points, $\tilde{S}$ dotted, S plain line.

Conclusion

We have seen

- A composite hyperbolic distortion
- With good fitting properties (estimation and convergence)
- Useful for stochastic simulations, due to invertibility property

Perspectives

- Choice of parameter number - forecasting
- Adaptation of some multiplicative or additive models
- Multivariate case - Dependencies
- Stochastic distortions

Thank you for your attention

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