Duality Methods to price financial derivatives that replicate portfolios compounds of tradable and non tradable assets.

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January,2011

Resume

We consider a one period model to valuate financial derivatives that replicate portfolios compounds of tradable and non tradable assets. This problem is solved by introducing a dual problem that needed the indifference price. In addition, classification of the utility functions with risk aversion coefficient is used to generalize results; also some numerical issues are solved using algorithms of speed up.

Market

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- Ω is a finite sample space, with $K < \infty$
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Definition A price process $S = \{S_t : t \in \tau\}$ where $S(t, w) := (S_1(t, w), \ldots, S_N(t, w))^T$, $N < \infty$ and $S_n(t, w)$ is price of security at the time t.

Portafolio

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Definition The value process, $V_t = \{V_t, t = 0, 1\}$ describes the total value of the portfolio at each point in time,

$$V_t(H) = B(t) + \sum_{j=1}^{N} H_j S_j(t) = B(t) + < H; S(t) > .$$

A bank account process $B(t)$ where $B(0) = 1$ and B_1 is a random variable.

Risk random probability

Definition A probability measure Q on Ω is said to be a risk neutral probability measure if $Q(w) > 0$ all $w \in \Omega$ and,

$$E_Q[S(1)] = \sum_{i=1}^{K} Q(w_i) S(1, w_i) = S(0).$$

$E_Q[X]$ means the expected value of random variable X under the probability measure Q.

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Definition Let $C_1 : \Omega \longrightarrow \mathbb{R}$ a contingent claim. C_1 is said to be attainable or marketable if there exists some trading strategy H, called the replicating portfolio, such that $V_1(H) = C_1$. In this case one says that H generates C_1.

Risk neutral valuation principle If the single period model has only one risk neutrak probability measure, then at the time $t = 0$, the price of an marketable contingent claim C_1 is $E_Q[X/B_1]$, where Q is any risk neutral probability measure.

$$p = E_Q[V_1^*(H)] = E_Q[C_1/B_1]$$

Complete and Incomplete Markets

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Theorem

The market is complete if and only if there is only one risk neutral probability measure

When there are no non-tradable assets and sufficiently many tradable assets to span the market one can invest to replicate future risk. The initial value of the portfolio must then equal the arbitrage free price of the future position. This method does not work when certain assets cannot be traded, giving rise to positions which cannot be hedged.

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To price such assets and related claims, we proceed by investigating optimal investment policies for different utility functions. The price introduced, the indifference price, equates the difference in the initial values of an optimal investment in the two cases when one can, or cannot, obtain some untradable claim at the final time.

Arrow- Debreu Model

The Arrow-Debreu model is introduced to rewrite the elements of the market so that we can solve future problems.
Let $\mathcal{F}_S = \sigma(S(1))$ and suppose $\{F_1; ...; F_m\} \subseteq \mathcal{F}_S$ are the atoms of $\mathcal{F}_S$. For each atom,

$$A_s = \{i \in \{1, ..., K\} | w_i \in F_s\},$$

where $1 \leq s \leq m$. This is called the partition of the states.

The securities of the Arrow-Debreu whem $w_i \in F_s$ some s such that $1 \leq s \leq m$, at time $t = 1$ are:

$$a_1^s(w_i) = \mathbb{I}_{F_s}(w_i) = \begin{cases} 1, & \text{si } i \in A_s \text{ ó } w_i \in F_s; \\ 0, & \text{si } i \text{ no está en } A_s. \end{cases}$$

and time $t = 0$,

$$a_0^s = E_{\mathbb{P}}(\mathbb{I}_{F_s}(w)) = q_s = \frac{\#(F_s)}{K}.$$

Consider $\sum_{s=1}^m a_0^s = 1$, and $\forall w$ c $\sum_{s=1}^m a_1^s(w_i) = \sum_{s=1}^m q_s = 1$.

We now characterize the risk neutral measures

Teorema

Let $Q : \Omega \longrightarrow \mathbb{R}$ risk neutral measures, if and only if there exists a random variable $n : \Omega \longrightarrow \mathbb{R}$ (denote $n(w_l) = n_l$), so that

$$Q(w_l) = Q_l = q_s e^{n_l} p_l, \tag{1}$$

$$\sum_{l \in A_s} p_l e^{n_l} = 1, \ \forall s \text{ tal que } 1 \leq s \leq m. \tag{2}$$

Where $q_s = \sum_{l \in A_s} Q_l$ para $1 \leq s \leq m$.

Utility Function.

Definition Let $U : \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$ a strictly concave, continuously differentiable function satisfying $U'(-\infty)$ and $U'(\infty)$.

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Example

- Exponential $U(x) = -e^{-\gamma x}$, $x \geq 0$
- Potencial $U(x) = \frac{1}{\alpha}x^{\alpha}$; $x \geq 0$ y $\alpha < -1$
- Logarithm: $U(x) = \ln(x)$;$x \geq 0$

Definición A useful quantity for the discussion of utilities is the coefficient of absolute risk aversion $R_a : \mathbb{R} \longrightarrow \mathbb{R}$

$$R_a(x) = -\frac{U''(x)}{U'(x)},$$

and a utility function is of the Hara class if $R_a(x)$ satisfies

$$R_a(x) = \frac{1}{A + Bx}, \tag{3}$$

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Definición The convex conjugate function $\tilde{U}$ of the utility function is the Legendre-Fenchel transform of the convex function $-U$, that is

$$\tilde{U}(y) := \max_{x>0}[U(x) - xy] = -\min_{x>0}[xy - U(x)]. \tag{4}$$

Transaction Cost

A transaction cost amounting to $G(v, S)$, where v is the volume of shares transacted (either positive or negative) and S is share price. For most purposes, we shall specialize this to the case of costs that are a constant proportion of the value transacted,

$$G(v, S) = k < v; S >= k(v^1 S_1 + ... + v^N S_N),$$

$$G(v, S) = k||v||_1 = k(|v^1| + ... + |v^N|),$$

where $||.||_1$ is the norm 1 over vector v.

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Let $\mathcal{H}$ set of all tranding strategy and x our initial wealth. We now describe the structure of the general problem. Using the notation already introduced, we define the value function

$$E_{\mathbb{P}}[U(V_1(H,x))].$$

We calculate replication strategies that maximize the expected value of this utility function.

$$\max\{E_{\mathbb{P}}\left[U(V_1(H,x))\right] | H \in \mathcal{H}, V_0 = x\}$$

It is tractable only in the case of exponential utility. Definite,

$$J_G(x) = \sup_{H\in\mathcal{H}} E_{\mathbb{P}}[U(V_1(H,x)-G)] = \sup_{H\in\mathcal{H}} \sum_{i=1}^{K} p_i U(V_1(H,x,w_i)-g_i),$$

$$J_0(x) = \sup_{H\in\mathcal{H}} E_{\mathbb{P}}[U(V_1(H,x))]$$

Indifference price

The utility indifference buy (or bid) price p is the price at which the investor is indifferent (in the sense that his expected utility under optimal trading is unchanged) between paying nothing and not having the claim X and paying v now to receive the claim X at time T.

$$J_G(x+v) = J_0(x).$$

To solve the above equation was necessary to use duality theory of Fenchel-Rockafellar developed the author Wilfred Kaplan.
This consiste in make a definition of dual price, ,

- $\tilde{J}_G(y) = \inf_{Q \in \mathcal{M}} \left(E_{\mathbb{P}} \left(\tilde{U} \left(\frac{Qy}{\mathbb{P}} \right) - yE_Q(G) \right) \right)$
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$$\tilde{J}_G(y) = \sup_x [J_G(x) - xy]$$

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This last theorem allows the calculation for the dual value, then this is returned to the original problem.

The procedure to obtain the indifferent price is,

1. Compute $J_0(x)$.
2. Compute $\hat{y}$ by that

$$\tilde{J}_G(\hat{y}) + (x+v)\hat{y} = \tilde{J}_G(\hat{y}) + f^G(\hat{y})\hat{y} = J_0(x)$$

   Where $f^G(\hat{y}) = x + v = \frac{1}{y}\sum_{s=1}^{k} \lambda_s^G(y) = -\tilde{J}'_G(y)$
3. Obtain $v = f^G(\hat{y}) - x$.

At each step of the previous algorithm is used the algorithm Newton-Rhapson applying acceleration methods that allow fast convergence.
As determinants of this model are the utility function and the Arrow-Debreu model.
Proving that the value of indifference price does not depend on utility function, but depends on how you choose the partition of the states with the Arrow-Debreu model.

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Thanks.