L'oscillateur harmonique lineaire et non-lineaire perturbé par un bruit blanc Gaussien et fréquence de Rice.

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Introduction

An oscillations model with many applications it is the harmonic oscillator perturbed with white noise. Its equation is the following

$$\ddot{X}_t = \sigma dW_t - (\gamma \dot{X}_t + \nabla V(X_t))dt.$$

Where W_t is a Brownian motion, γ is a positive constant and V is the potential.
By using the change of variable $\dot{X}_t = Y_t$ the above equation is equivalent to the system of SDE

$$\begin{aligned} dX_t &= Y_t dt \\ dY_t &= \sigma dW_t - (\gamma Y_t + \nabla V(X_t))dt \end{aligned} \qquad (1)$$

When $\sigma = 0$ the system is dissipative because the existence of the damping force $-\gamma y$ and it converges to the phase points where the Hamiltonian $H(x, y) = \frac{1}{2}y^2 + V(x)$ attains the local minima. When $\sigma > 0$ the random force will compensate the loss of energy and the system will approach to a unique invariant probability measure.

$$\mu(dx, dy) = \frac{1}{C} e^{-\frac{2\gamma}{\sigma^2} H(x,y)}.$$

In this talk we are interested in study the Rice's frequency of these oscillators that can be defined as

$$< \omega >_R = \lim_{t\to\infty} \frac{N_t^X(0)}{t},$$

where $N_t^X(0) = \#\{s \leq t \, : X_s = 0\}$ is the number of zero crossings of the process X.

Two potentials will be of our interest. The quadratic potential $V(x) = \frac{\omega_0^2 x^2}{2}$, and the Duffing's potential $V(x) = \frac{x^4}{4} - \frac{x^2}{2}$. In the first case the solution of system (1) is a Gaussian process having as invariant measure

$$\mu(dx, dy) = \frac{\gamma\omega_0}{\pi\sigma^2} e^{-\frac{\gamma}{\sigma^2}(\omega_0^2 x^2 + y^2)} dxdy.$$

Thus for all fixed $t \geq 0$, in the stationary regimen, the coordinates of the vector (X_t, Y_t) are independent and moreover

$$\mathcal{L}(X_t) = N(0, \frac{\sigma^2}{2\gamma\omega^2}) \text{ and } \mathcal{L}(Y_t) = N(0, \frac{\sigma^2}{2\gamma}).$$

In the second case the invariant probability measure is

$$\mu(dx,dy)=\frac{\sqrt{\gamma}}{\sqrt{\pi}\sigma C}e^{-\frac{2\gamma}{\sigma^2}(\frac{x^4}{4}-\frac{x^2}{2}+\frac{y^2}{2})}dxdy,$$

where $C(\gamma,\sigma^2):=\int_{\mathbb{R}}e^{-\frac{2\gamma}{\sigma^2}(\frac{x^4}{4}-\frac{x^2}{2})}dx.$

Hence here, also under stationary regimen, the vector (X_t,Y_t) has independent coordinates and

$$p_{X_t}(x)dx=\frac{1}{C(\gamma,\sigma^2)}e^{-\frac{2\gamma}{\sigma^2}(\frac{x^4}{4}-\frac{x^2}{2})}dx \text{ and}$$

$\mathcal{L}(Y_t)=N(0,\frac{\sigma^2}{2\gamma}).$

Rice's frequency and ergodicity.

If the stationary mean zero process X has a continuous derivative Y and the following conditions are satisfied

1. Function $x \to p_{X_0}(x)$ is continuous for x in a neighborhood of 0.
2. $(x, y) \to p_{X_0,Y_0}(x, y)$ is continuous for x in a neighborhood of 0 and $y \in \mathbb{R}$.
3. Let define $A_1(u, x) = \int_{\mathbb{R}^2} |y_1 - y| p_{X_0,Y_0,Y_u}(x, y, y_1) dy dy_1$. If $A_1(u, x)$ tends to zero as $u \to 0$ uniformly, for x in a neighborhood of 0.

The following Rice's formula holds

$$\mathbb{E}N_t^X(0) = t p_{X_0}(0)\mathbb{E}|\dot{X}_0|.$$

In the quadratic potential case the process is stationary and Gaussian. Moreover the precedent conditions are easily verified. And we obtain

$$\mathbb{E}N_t^X(0) = t\frac{\omega_0}{\pi}$$

Let us remark that this quantity only depends on ω_0. The natural frequency of the oscillator.

For the Duffing's potential, the system of SDO is known as the Kramer's oscillator, the conditions (1) and (2) hold. The condition (3) is more involved. We will give a sketch of the proof. Our inspiration will be the very interesting article of Liming Wu:

" Large and moderate deviations and exponential convergence for stochastic damping Hamiltonian system". SPA (2001).

The process (X_t, Y_t) generates a Markovian strong Feller semigroup P_t. Its action over the space $C_b(\mathbb{R})$ of continuous and bounded functions can be expressed via the Girsanov formula.

$$P_t(f)(x,y) = \int_{\mathbb{R}^2} p_t(x,y,x_1,y_1) f(x_1,y_1) dx_1 dy_1 = \mathbb{E}^{(x,y)}[f(X_t, Y_t)]$$
$$= \mathbb{E}[M(t) f(x + yt + \sigma \int_0^t W_s ds, y + \sigma W_t)]$$

where W_t is the standard unidimensional Brownian Motion and

$$M(t) =$$
$$exp\Big(- \frac{1}{\sigma} \int_0^t [\gamma Y_s + \nabla_x (X_s)] dW_s - \frac{1}{2\sigma^2} \int_0^t [\gamma Y_s + \nabla_x V(X_s)]^2 ds \Big).$$

$Y_s = y + \sigma W_s$ and $X_s = x + ys + \sigma \int_0^s W_u du$.
Wu also shows that the domain of P_t can be extended to some non bounded functions.

In particular for the functions $G_{y_1}(x', y') := |y' - y_1|$. Thus

$$P_t(G_{y_1})(x,y) = \mathbb{E}[M(t)G_{y_1}(x + yt + \sigma \int_0^t W_s ds, y + \sigma W_t)]$$

$$|P_t(G_{y_1})(x,y) - G_{y_1}(x,y)| \leq \sigma \mathbb{E}[M(t)|W_t|] \leq C\sigma \sqrt{t \log \frac{1}{t}},$$

by Levy's modulus of continuity.
We can write

$$A_1(s,x) := p_1(x) \int_{\mathbb{R}} P_s(G_y)(x,y) p_2(y) dy.$$

And the result follows because $G_y(x,y)=0$. Hence the Rice's formula holds for this model and we get

$$\mathbb{E}N_t^X(0)=t\frac{1}{C(\gamma,\sigma^2)}\frac{\sigma}{\sqrt{\pi\gamma}}.$$

What about the limits.

1.- Case of the quadratic potential. The spectral density of the stationary solution is $f(\lambda)=\frac{\sigma^2}{2\pi[(\lambda^2-\omega_0^2)^2+\gamma^2\lambda^2]}$. As $f>0$ the

process is ergodic, and the Ergodic Theorem yields:

$$<\omega>_R=\lim_{t\to\infty}\frac{N_t^X(0)}{t}=\frac{\omega_0}{\pi}\,\text{a.s.}$$

This result allows building an estimator of the natural frequency of the oscillator.

2.- Case of the Duffing potential. In the Wu article is also shown that process (X_t, Y_t) is exponentially ergodic in some class of continuous functions. This result entails

$$< \omega >_R = \lim_{t\to\infty} \frac{N_t^X(0)}{t} = \frac{1}{C(\gamma,\sigma^2)} \frac{\sigma}{\sqrt{\pi\gamma}} \text{ a.s.}$$

The precedent two results can be viewed as a possibility for defining the frequency in random oscillations. In what follows we will explore this concept.

CLT and frequency estimation

In the case of quadratic potential, we have defined an a.s convergent estimator for the natural frequency of the harmonic oscillator. Hence, we can look for a confidence interval for such an estimator. Our plan consists in two steps.

- **Step one.** Prove a CLT for the quantity $\sqrt{t}[\frac{N_t^X(0)}{t} - \frac{\omega_0}{\pi}]$.
- **Step two.** Build an estimator for the variance limit.

For the first step we must prove first that the second moment for the r.v. $N_t^X(0)$ is finite. This is a consequence of the Geman's condition: If r denotes the covariance function of the stationary Gaussian process X

$$\mathbb{E}(N_t^X(0))^2 < \infty \Leftrightarrow \int_0^\delta \frac{r''(t) - r''(0)}{t} dt < \infty.$$

An elementary computation gives

$$\int_0^\delta \frac{r''(t) - r''(0)}{t} dt <$$
$$Const \int_0^\infty |\log(\lambda\delta)| \frac{\sigma^2}{2\pi[(\lambda^2 - \omega_0^2)^2 + \gamma^2\lambda^2]} d\lambda < \infty.$$

In a joint work (SPA 1997) with Marie Kratz, we have shown that the following expansion holds true (let define $m_0 = \frac{\sigma^2}{2\gamma\omega_0^2}$, $m_2 = \frac{\sigma^2}{2\gamma}$, $b_{2k} = \frac{H_{2k}(0)}{(2k)!}$ and $a_{2l} = \frac{(-1)^{l+1}}{2^l l!(2l-1)}$)

$$\sqrt{t}(\frac{N_t^X(0)}{t} - \frac{\omega_0}{\pi}) =$$

$$= \frac{\omega_0}{\pi} \sum_{q=2}^{\infty} \sum_{l=0}^{q} b_{2(q-l)} a_{2l} \frac{1}{\sqrt{t}} \int_0^t H_{2(q-l)}(\frac{X_s}{\sqrt{m_0}}) H_{2k}(\frac{Y_s}{\sqrt{m_2}}) ds.$$

This expression can be written

$\sqrt{t}(\frac{N_t^X(0)}{t} - \frac{\omega_0}{\pi}) = \frac{\omega_0}{\pi} \sum_{q=1}^{\infty} I_q^t(f_{2q})$.

For a sequence f_q of $L^2(\mathbb{R}^q, dx)$ symmetric functions and $I_q(\cdot)$ denotes an element of the q-nth Itô-Wiener chaos. In fact

$$I_q^t(f_{2q}) = \frac{1}{\sqrt{t}} \int_0^t \sum_{l=0}^{q} b_{2(q-l)} a_{2l} H_{2(q-l)}(\frac{X_s}{\sqrt{m_0}}) H_{2k}(\frac{Y_s}{\sqrt{m_2}}) ds.$$

Let us recall that, if B is a complex Brownian motion, we have

$X_t = \int_{-\infty}^{\infty} e^{it\lambda} \sqrt{f(\lambda)} dB(\lambda)$ and $Y_t = \int_{-\infty}^{\infty} e^{it\lambda} i\lambda \sqrt{f(\lambda)} dB(\lambda)$.
We will use an approximation for a m-dependent process i.e.

$$X_t^m = \int_{-\infty}^{\infty} e^{it\lambda} \sqrt{f * \hat{\varphi}_m(\lambda)} dB(\lambda),$$

where $\hat{\varphi}_m(\lambda) = m\hat{\varphi}(m\lambda)$, φ is an even positive real function with support in $[-1/2, 1/2]$ and such that $||\varphi||_2 = 1$. The process X^m is m-dependent.

By defining

$$I_q^t(f_{2q}(m)) = \frac{1}{\sqrt{t}} \int_0^t \sum_{l=0}^q b_{2(q-l)} a_{2l} H_{2(q-l)}(\frac{X_s^m}{\sqrt{m_0}}) H_{2k}(\frac{Y_s^m}{\sqrt{m_2}}) ds.$$

We show that

$$\lim_{m\to\infty} \lim_{t\to\infty} \mathbb{E}(I_q^t(f_q) - I_q^t(f_q(m)))^2 = 0.$$

Thus the asymptotic normality of $I_q^t(f_q)$ follows by the CLT for m-dependent random variables and the above approximation. To finish the proof we must show that the limit variance of the $\sqrt{t}(\frac{N_t^X(0)}{t} - \frac{\omega_0}{\pi})$ exists and is finite. This can be made applying the Arcones inequality.

Hence we get

$$\gamma^2 := \lim_{t\to\infty} tVar(\frac{N_t^X(0)}{t} - \frac{\omega_0}{\pi}) =$$

$2\frac{\omega_0^2}{\pi^2}\sum_{q=1}^{\infty}\int_0^{\infty}\mathbb{E}[I_q^0(f_{2q})I_q^t(f_{2q})]dt < \infty$. The results follows because:

1. The infinite series defining $\sqrt{t}(\frac{N_t^X(0)}{t} - \frac{\omega_0}{\pi})$ is truncated.
2. Then the CLT is obtained for the finite part, given the asymptotic independence of each $2q$-limit.
3. Finally by using the $L^2(\Omega)$ proximity.

Variance estimation

The limit variance obtained in the last section is written as an infinite sum. Hence it is impossible or very difficult to compute. We will propose a method for its estimation. We will adapt to our particular case the procedure provided in a joint work with P. Doukhan and J. Jakubowicz. In that follows we consider $t = n \in \mathbb{N}$ and we put $Y_j = N^X_{j+1}(0) - N^X_j(0)$. Hence

$$\sqrt{n}\left(\frac{N^X_n(0)}{n} - \mathbb{E}[N^X_n(0)]\right) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (Y_j - \mathbb{E}Y_j).$$

We must consider that we observe either $Z_j = Y_j - \mathbb{E}Y_j$ or that we have obtained a weakly independent estimation of $\mathbb{E}Y_j$. For sake of simplicity we will work with the first of the options.

We split the interval of integers $[1, n]$ in several subintervals. Defining $l = l(n)$, $m = m(n)$, $\tilde{m} = m + l$, $N = N(n)$ with $n \geq N(m+l) - l$.
Let define

$$\Delta_{i,m} = \frac{1}{\sqrt{m}} \sum_{j=(i-1)(m+l)}^{(i-1)(m+l)+m} Z_j.$$

And putting $\hat{\gamma}_n^2 := \frac{1}{N(n)} \sum_{i=1}^{N(n)} (\Delta_{i,m})^2$.

Our process X is α-mixing with an exponential coefficient. This implies that we can obtain by using classical moment inequalities the bound $Var(\hat{\gamma}_n^2) \leq \frac{C}{N}$. Therefore the LLN holds and we conclude

$\hat{\gamma}_n^2 \to \gamma^2$ in L^2.

Remark: Note that we have shown $\mathbb{E}[\hat{\gamma}_n^2 - \mathbb{E}\hat{\gamma}_n^2] \to 0$. The LLN follows by bounding the rest of the series that defines the variance by using a covariance inequality.

We show below two graphs containing the simulation of this process. In first place we take the time of the system equals 100

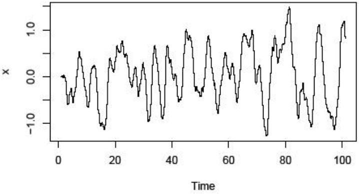


Figure: Simulation of the linear oscillator $V(x) = \omega_0^2 \frac{x^2}{2}$ $\omega_0^2 = 1$, $\gamma = 1$ and $\sigma = 1$. $T = 100$

Here we take $T = 1000$.

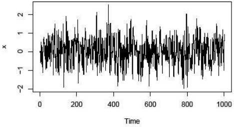


Figure: Simulation of the linear oscillator $V(x) = \omega_0^2 \frac{x^2}{2}$ $\omega_0^2 = 1$, $\gamma = 1$ and $\sigma = 1$. $T = 1000$

First we illustrate with a graph of the process $N_t^X(0)$.

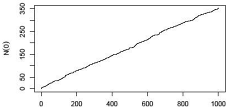


Figure: $N_t^X(0)$, $t \leq 1000$

and then we have the estimation of the frequency with $\omega_0 = 1$.

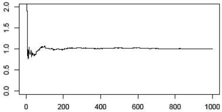


Figure: Estimation of ω_0, $T = 1000$

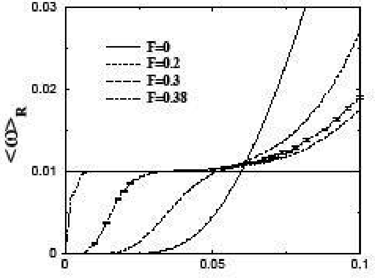
By using our method we obtain a confidence interval for γ^2 of 0.03 ± 0.01. Nevertheless, for obtaining this confidence interval we need to have a CLT for the following sequence $\sqrt{N(n)}(\hat{\gamma}_n^2 - \gamma^2)$ this result was also shown in the DJL paper.

Some reflections about the Kramers oscillator.
The Rice's frequency for the Kramers oscillator is interesting for studying the synchronization phenomenon.

Indeed simulation studies have shown that Kramers oscillator excited with a periodical signal $Z_t = F\cos(\Omega t)$ (i.e. the equation reads now)

$$\ddot{X}_t = \sigma dW_t - (\gamma \dot{X}_t + \nabla V(X_t))dt + F\cos(\Omega t),$$

for a certain range of values of σ synchronize it Rice´s frequency with the one of the external signal. This problem has only been considered in the physics literature by using simulation studies and it has not been studied in a formal mathematical framework. Below we show a graph showing a result of these simulations (taken of [4]).



Numerically determined Rice frequencies of the periodically driven bistable Kramers oscillator Eq. (76) computed with the friction coefficient $\gamma = 0.5$ and the angular driving frequency $\Omega = 0.01$ and plotted as a function of the noise intensity . Different curves correspond to various amplitudes of the harmonic drive F. For larger values of F wider regions appear where the Rice frequency is locked to the external driving frequency Ω.

In the linear case in a joint work with M. Kratz (2009) we were able to prove a CLT for the Rice's frequency for the process solution of such equation in the regime of large t.
This was a simple consequence of the linearity because the crossings of zero of the solution Y of the above equation are simply the crossings of a periodic function. The CLT was proven in the similar way that the one that was indicated in this talk.

The problem for the Kramers oscillator even if $F \equiv 0$, seem hard and requires to develop new technics.

Program of research

- First we will study the properties of weak dependance for the process (X_t, Y_t), stationary solution of the system of SDE.
- It will be interesting to define an estimator of the density of the invariant measure for this last process, by using a set of observations in a uniform mesh. There exists a difficulty inherent of this data: one observes only X_t, then the derivative Y_t must be estimated by finite differences.
- It would be important to prove, for the crossings of the process X_{t_k}, that they satisfy a LLN and also a CLT.
- The problem of estimate the potential is similar of the estimation of a regression function, but that matter could be more involved that the three problems mentioned before.

We based this talk on the work made by me and my coworkers: P. Doukhan, J. Jakubowicz, M. Kratz and in a work in progress "avec Clémentine Prieur".

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