Plug-in estimation of level sets in a non-compact setting

Elena Di Bernardino[a] joint work with Thomas Laloë[b]

Dijon, Journées Ast&risk 17-18 January 2011

[a] Université Lyon 1, ISFA, Laboratoire SAF
[b] Université de Nice Sophia-Antipolis, Laboratoire J-A Dieudonné

Framework

Goal: In this paper, we consider the problem of estimating the level sets of a bivariate distribution function F. More precisely our goal is to build a consistent estimator of

$$L(c) := \{F(x) \geq c\}, \quad \text{for } c \in (0,1).$$

Idea: We consider a *plug-in* approach that is $L(c)$ is estimated by

$$L_n(c) := \{F_n(x) \geq c\}, \quad \text{for } c \in (0,1),$$

where F_n is a consistent estimator of F.

Tools and general aspects

- We do not suppose any compactness property for the level sets we estimate. This requires special attention in the statement of the problem.
- We state consistency results with respect to two proximity criteria between sets: the Hausdorff distance and the volume of the symmetric difference.

Our results are motivated by *applications in bivariate risk theory*. In this sense we also present simulated and real data examples which illustrate our theoretical results.

Key words: Level sets, distribution function, plug-in estimation, Hausdorff distance, Conditional Tail Expectation.

Literature and background

Estimation of the level sets of a density function: Polonik (1995), Tsybakov (1997), Baíllo *et al.* (2001)

Estimation of the level sets of a regression function in a compact setting: Cavalier (1997), Biau *et al.* (2007), Laloë (2009)

Estimation of general compact level sets: Cuevas *et al.* (2006)

An alternative approach, based on the geometric properties of the compact support sets: Cuevas and Fraiman (1997), Cuevas and Rodríguez-Casal (2004).

Contents

Contents

Notation and preliminaries

Let ${\mathbb{R}_+^2}^* := \mathbb{R}_+^2 \setminus (0,0)$, $\mathcal{F}$ the set of continuous distribution functions $f : \mathbb{R}_+^2 \to [0,1]$ and $F \in \mathcal{F}$.

Given an *i.i.d* sample $\{X_i\}_{i=1}^n$ in $\mathbb{R}_+^2$ with distribution function F, we denote $F_n(.) = F_n(X_1, X_2, \ldots, X_n, .)$ an estimator of F.

Define, for $c \in (0,1)$, the upper c-level set of $F \in \mathcal{F}$ and its *plug-in estimator*:

$$L(c) := \{x \in \mathbb{R}_+^2 : F(x) \geq c\}, \quad L_n(c) := \{x \in \mathbb{R}_+^2 : F_n(x) \geq c\}.$$

In addition, given $T > 0$, we set

$$L(c)^T := \{x \in \mathbb{R}_+^2 \cap [0, T]^2 : F(x) \geq c\},$$

$$L_n(c)^T := \{x \in \mathbb{R}_+^2 \cap [0, T]^2 : F_n(x) \geq c\},$$

and, for any $A \subset \mathbb{R}_+^2$, we note ∂A its boundary.

Notation and preliminaries

We now introduce the class of quasi concave functions:

Definition

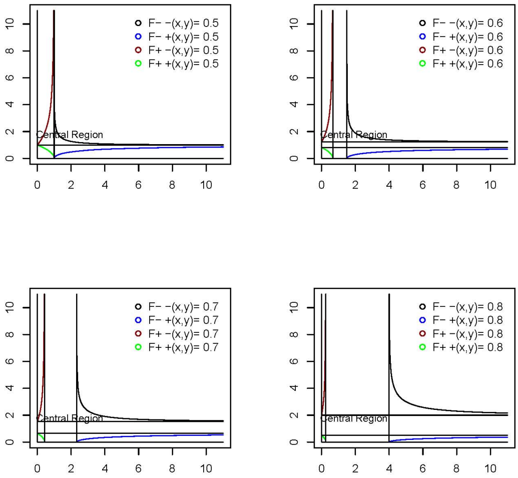
A function $F \in \mathcal{F}$ is quasi concave if it satisfies one of the following equivalent conditions:

1. $\forall\ x, y\ \in \mathbb{R}^2_+$ and $a \in (0,1)$, $F(ax + (1-a)y) \geq \min\{F(x), F(y)\}$.
2. The upper level sets of F are convex.

.

If F is a quasi concave bivariate distribution function on $\mathbb{R}^2_+$ then each $\partial L(c)$, for $c \in (0,1)$, is identified by a monotone decreasing and convex curve in the plane $\mathbb{R}^2_+$ (Rossi, 1973; Tibiletti, 1991, 1994, 1995).

From a parametric formulation of $\partial L(c)$ (see Belzunce *et al.* 2007).
Case: Survival Clayton Copula with marginals (Burr(1), Burr(2))



Notation and preliminaries

For $r > 0$ and $\lambda > 0$, define

$$E := B(\{x \in \mathbb{R}_+^2 : \mid F - c \mid \leq r\}, \lambda),$$

$$m^{\nabla} := \inf_{x \in E} \|(\nabla F)_x\|, \qquad M_H := \sup_{x \in E} \|(HF)_x\|,$$

- $B(x, \xi)$ is the closed ball centered on x and with positive radius ξ,
- $(\nabla F)_x$ is the gradient of F evaluated at x,
- $\|(HF)_x\|$ is the matrix norm induced by Euclidean distance of the Hessian matrix in x.

Hausdorff distance between A_1 and A_2 is defined by

$$d_H(A_1, A_2) = \inf\{\varepsilon > 0 : A_1 \subset B(A_2, \varepsilon), A_2 \subset B(A_1, \varepsilon)\},$$

where $B(S, \varepsilon) = \bigcup_{x \in S} B(x, \varepsilon)$; A_1 and A_2 are compact sets in $(\mathbb{R}_+^2, d)$.

Assumption H

Finally, we introduce the following assumption (e.g. see Tsybakov, 1997; Cuevas *et al.*, 2006):

H: There exist $\gamma > 0$ and $A > 0$ such that if $|t - c| \leq \gamma$ then

$$d_H(\partial L(c)^T, \partial L(t)^T) \leq A|t - c|.$$

.

Sufficient condition for Assumption **H** can be obtained in terms of the differentiability properties of F.

Contents

About Assumption H

H is satisfied under mild assumptions (modification of Proposition 3.1 in the PhD of Rodríguez-Casal, 2003).

Proposition

Let $F \in \mathcal{F}$ quasi-concave on $\mathbb{R}^2_+$ and twice differentiable on $\mathbb{R}^{2*}_+$. Let $T > 0$ such that $\partial L(c)^T$ is a non-empty set. Assume there exist $r > 0$, $\lambda > 0$ such that $m^\nabla > 0$ and $M_H < \infty$. Then F satisfies Assumption **H**, *with $A = \frac{2}{m^\nabla}$.*

From the quasi-concave property we know the form of $\partial L(c)$, so there exists $T > 0$ such that $\partial L(c)^T \neq \emptyset$. For all $\lambda > 0$ and $r > 0$:

$$\{x \in \mathbb{R}^2_+ : \mid F - c \mid \leq r\} \neq \emptyset, \quad E \neq \emptyset,$$

because of the continuity of F.

Consistency in terms of the Hausdorff distance

From now on we note $\|F - F_n\|_\infty = \sup_{x \in \mathbb{R}_+^2} \mid F(x) - F_n(x) \mid$, and for $T > 0$ the truncated version $\|F - F_n\|_\infty^T = \sup_{x \in [0,T]^2} \mid F(x) - F_n(x) \mid$.

Theorem (Consistency Hausdorff distance)

Let $F \in \mathcal{F}$ quasi-concave on $\mathbb{R}_+^2$ and twice differentiable on $\mathbb{R}_+^{2*}$. Assume that there exists $r > 0$, $\lambda > 0$ such that $m^\nabla > 0$ and $M_H < \infty$. Let $T_1 > 0$ such that $\partial L(c)^{T_1}$ is a non-empty set, and $(T_n)_{n \in \mathbb{N}^*}$ an increasing sequence of positive values. Assume that, for each n, F_n is continuous with probability one and

$$\|F - F_n\|_\infty \to 0, \quad a.s.$$

Then

$$d_H(\partial L(c)^{T_n}, \partial L_n(c)^{T_n}) \leq 6\,A\,\|F - F_n\|_\infty^{T_n}, \quad a.s.,$$

with $A = \frac{2}{m^\nabla}$.

Consistency in terms of the Hausdorff distance

From Theorem *Consistency Hausdorff distance* we can trivially deduce the following result:

Corollary

Under assumptions of Theorem Consistency Hausdorff distance,

$$d_H(\partial L(c)^{T_n}, \partial L_n(c)^{T_n}) \leq 6\, A \, \|F - F_n\|_\infty, \quad a.s.,$$

with $A = \frac{2}{m^\nabla}$.

- $d_H(\partial L(c)^{T_n}, \partial L_n(c)^{T_n})$ converges to zero at least at the same rate as $\|F - F_n\|_\infty$.
- Asymptotic result for a fixed level c. Case c close to one the constant A could be large (we will need a large number of data to get a reasonable estimation).

L_1 consistency

Consistency criterion: the consistency of the volume (in the Lebesgue measure sense) of the symmetric difference between $L(c)^{T_n}$ and $L_n(c)^{T_n}$.

We define the distance between two subsets A_1 and A_2 of $\mathbb{R}_2^+$ by

$$d_\lambda(A_1, A_2) = \lambda(A_1 \triangle A_2),$$

where λ stands for the Lebesgue measure on $\mathbb{R}^2$ and $\triangle$ for the symmetric difference.

Assumptions **A1**:

A1 There exists a positive increasing sequence v_n such that

$$v_n \, \|F - F_n\|_\infty \underset{n \to \infty}{\to} 0, \quad a.s.$$

L_1 consistency

Theorem (Consistency volume)

Let $F \in \mathcal{F}$ quasi-concave on $\mathbb{R}^2_+$ and twice differentiable on $\mathbb{R}^{2*}_+$. Assume that there exists $r > 0$, $\lambda > 0$ such that $m^\nabla > 0$ and $M_H < \infty$. Assume that for each n, F_n is continuous with probability one and that Assumption **A1** *is satisfied. Then for any increasing positive sequence $(T_n)_{n\in\mathbb{N}^*}$ such that $L(c)^{T_1}$ is a non-empty set and $T_n = o(v_n)$, holds*

$$d_\lambda(L(c)^{T_n}, L_n(c)^{T_n}) \underset{n\to\infty}{\to} 0, \quad a.s.$$

- Choice of the sequence T_n to ensure the convergence of $d_\lambda(L(c)^{T_n}, L_n(c)^{T_n})$, in terms of the convergence rate v_n.
- Assumptions satisfied for Independent copula and exponential marginals, Farlie-Gumbel-Morgenstern (FGM) or Survival Clayton copulas and Burr marginals, ...

Contents

Bivariate Value-at-Risk

The Value-at-Risk (VaR) measure is defined as

$$\mathrm{VaR}_\alpha(X) = Q_X(\alpha), \forall\, \alpha \in (0,1),$$

with $Q_X(\alpha) = F_X^{-1}(\alpha)$ (univariate quantile function of the continuous loss distribution function F_X).

Definition (Generalization of the VaR measure in dimension 2)

For $\alpha \in [0,1]$ and $F \in \mathcal{F}$, the bi-dimensional Value-at-Risk at probability level α is the boundary of its α-level set, i.e. $\mathrm{VaR}_\alpha(F) = \partial L(\alpha)$, with $\alpha \in (0,1)$.

References: Embrechts and Puccetti (2006), Tibiletti (1993) and Nappo and Spizzichino (2006). In Tibiletti (1993) with $\alpha = \frac{1}{2}$: natural extension of the bi-dimensional median.

Bivariate Value-at-Risk

From Belzunce *et al.* (2007) we can introduce the following definition:

Definition

Let $\alpha \in (0,1)$ and $F \in \mathcal{F}$. We define

1. *lateral region* $L_{(X,Y)}(\alpha) := \{(x,y) \in \mathbb{R}^2_+ : F(x,y) > \alpha\}$;
2. *estimated lateral region* $\widehat{L}_{(X,Y)}(\alpha) := \{(x,y) \in \mathbb{R}^2_+ : F_n(x,y) > \alpha\}$.

Note that $L_{(X,Y)}(\alpha) = L(\alpha) \backslash \partial L(\alpha)$.

Parametric formulation of $\partial L(\alpha)$ see Belzunce *et al.* (2007). Properties of the lateral regions see for instance Férnandez-Ponce and Suárez-Lloréns (2002) and Nappo and Spizzichino (2006).

Bivariate Value-at-Risk

Our (plug-in) estimator of the bivariate Value-at-Risk:

$$\mathrm{VaR}_\alpha(F_n) := \partial L_n(\alpha).$$

Consistency result for the $\mathrm{VaR}_\alpha(F_n)$, with respect to the Hausdorff distance:

$$d_H(\mathrm{VaR}_\alpha(F)^{T_n}, \mathrm{VaR}_\alpha(F_n)^{T_n}) \leq 6\,A\,\|F - F_n\|_\infty, \quad a.s.,$$

with $A = \frac{2}{m^\nabla}$.

VaR_α does not give any information about the thickness of the upper tail of the distribution function. Shortcoming of the VaR: we are not only concerned with the frequency of the default but also with the severity of loss in case of default.

In order to overcome this problem: Conditional Tail Expectation (CTE).

Bivariate Conditional Tail Expectation

For a continuous loss distribution function F_X the CTE at level α is defined by

$$CTE_\alpha(X) = \mathbb{E}[\, X \,|\, X > \mathrm{VaR}_\alpha(X)\,],$$

where $\mathrm{VaR}_\alpha(X)$ is the univariate VaR defined above (e.g. for properties of the $CTE_\alpha(X)$ see Denuit *et al.*, 2005).

Several bivariate generalizations of the classical univariate CTE:

- $\mathbb{E}[\,(X,Y)\,|\, X+Y > t\,]$,
- $\mathbb{E}[\,(X,Y)\,|\, \min\{X,Y\} > t\,]$,
- $\mathbb{E}[\,(X,Y)\,|\, \max\{X,Y\} > t\,]$,

e.g see Cai and Li, (2005).

Conditioning events are the total risk or some univariate extreme risk in the portfolio (aggregate variable).

A new bivariate Conditional Tail Expectation

Definition (Generalization of the CTE in dimension 2)

Consider an absolutely continuous positive random vector (X, Y) (with respect to the Lebesgue measure λ on $\mathbb{R}^2$) with bounded density $f_{X,Y}(x, y)$, $\mathbb{E}(X) < \infty$, $\mathbb{E}(Y) < \infty$ and $F \in \mathcal{F}$. For $\alpha \in (0, 1)$, we define

- the bivariate α-Conditional Tail Expectation

$$CTE_\alpha(X,Y) = \left\{ \begin{array}{l} \mathbb{E}[\, X \,|\, (X,Y) \in L_{(X,Y)}(\alpha) \,] \\ \mathbb{E}[\, Y \,|\, (X,Y) \in L_{(X,Y)}(\alpha) \,] \end{array} \right. .$$

- the estimated bivariate α-Conditional Tail Expectation

$$\widehat{CTE}_\alpha(X,Y) = \left\{ \begin{array}{l} \dfrac{\sum_{i=1}^n X_i 1_{\{(X_i,Y_i)\in \widehat{L}_{(X,Y)}(\alpha)\}}}{\sum_{i=1}^n 1_{\{(X_i,Y_i)\in \widehat{L}_{(X,Y)}(\alpha)\}}} \\ \dfrac{\sum_{i=1}^n Y_i 1_{\{(X_i,Y_i)\in \widehat{L}_{(X,Y)}(\alpha)\}}}{\sum_{i=1}^n 1_{\{(X_i,Y_i)\in \widehat{L}_{(X,Y)}(\alpha)\}}} \end{array} \right. .$$

A new bivariate Conditional Tail Expectation

Remarks:

- This bivariate Conditional Tail Expectation is a natural extension of the univariate one.
- If X and Y are identically distributed with a symmetric copula then $\mathbb{E}[X \mid (X,Y) \in L_{(X,Y)}(\alpha)] = \mathbb{E}[Y \mid (X,Y) \in L_{(X,Y)}(\alpha)]$.
- Our *geometric* $CTE_\alpha(X,Y)$ does not use an aggregate variable in order to analyze the bivariate risk's problem. $CTE_\alpha(X,Y)$ deals with the simultaneous joint damages considering the bivariate dependence structure of data in a specific risk's area of the plane $(L_{(X,Y)}(\alpha))$.

A new bivariate Conditional Tail Expectation

Let $\alpha \in (0,1)$ and $F \in \mathcal{F}$. Truncated version of the lateral regions:

$$L_{(X,Y)}(\alpha)^T := \{(x,y) \in \mathbb{R}_+^2 \cap [0,T]^2 : F(x,y) > \alpha\},$$
$$\widehat{L}_{(X,Y)}(\alpha)^T := \{(x,y) \in \mathbb{R}_+^2 \cap [0,T]^2 : F_n(x,y) > \alpha\}.$$

and of the theoretical and estimated CTE_α:

$$CTE_\alpha^T(X,Y) := \mathbb{E}[(X,Y) | (X,Y) \in L_{(X,Y)}(\alpha)^T],$$

$$\widehat{CTE}_\alpha^T(X,Y) = \begin{cases} \dfrac{\sum_{i=1}^n X_i 1_{\{(X_i,Y_i)\in \widehat{L}_{(X,Y)}(\alpha)^T\}}}{\sum_{i=1}^n 1_{\{(X_i,Y_i)\in \widehat{L}_{(X,Y)}(\alpha)^T\}}} \\ \dfrac{\sum_{i=1}^n Y_i 1_{\{(X_i,Y_i)\in \widehat{L}_{(X,Y)}(\alpha)^T\}}}{\sum_{i=1}^n 1_{\{(X_i,Y_i)\in \widehat{L}_{(X,Y)}(\alpha)^T\}}} \end{cases}.$$

A new bivariate Conditional Tail Expectation

Theorem (Consistency of $\widehat{CTE}_\alpha(X,Y)$)

Assume that (X,Y) is an absolutely continuous positive random vector (with respect to the Lebesgue measure λ on $\mathbb{R}^2$) with bounded density $f_{X,Y}(x,y)$, $\mathbb{E}(X) < \infty$, $\mathbb{E}(Y) < \infty$ and all the assumptions of Theorem Consistency volume are satisfied. Then

$$CTE_\alpha^{T_n}(X,Y) - \widehat{CTE}_\alpha^{T_n}(X,Y) \underset{n\to\infty}{\to} 0, \quad a.s. \tag{1}$$

The convergence in (1) must be interpreted as a componentwise convergence.

From (1) we can trivially deduce the following corollary:

Corollary

$$CTE_\alpha(X,Y) - \widehat{CTE}_\alpha^{T_n}(X,Y) \underset{n\to\infty}{\to} 0, \quad a.s.$$

Contents

Estimation of the level sets

Considering a grid of 10000 points, we give the percentage of points (averaged on 25 iterations) falling in the symmetric difference $L \triangle L_n$, for $\alpha \in \{0.1, 0.24, 0.38, 0.52, 0.66, 0.8\}$.

α	n= 500	n=1000	n=1500	n=2000
0.10	0.328	0.134	0.122	0.156
0.24	0.636	0.215	0.234	0.335
0.38	0.746	0.486	0.361	0.303
0.52	0.999	0.539	0.475	0.325
0.66	0.979	0.649	0.674	0.342
0.80	1.482	0.874	0.856	0.417

Table: Distribution with independent and exponentially distributed marginals with parameter $\lambda = 1$ and $\lambda = 2$ respectively.

Estimation of the level sets

α	n= 500	n=1000	n=1500	n=2000
0.10	0.113	0.079	0.072	0.002
0.24	0.089	0.040	0.041	0.001
0.38	0.152	0.115	0.119	0.001
0.52	0.204	0.168	0.082	0.077
0.66	0.216	0.271	0.199	0.186
0.80	0.603	0.236	0.181	0.115

Table: Distribution with Survival Clayton copula with parameter 1 and Burr marginals with parameter 2.

As expected, the greater n is, the better the estimations are. Moreover we note that for big values of α we need more data to get a good estimation of the level sets. This may come from the fact that when α grows the gradient of the distribution function decreases to zero and the constant A grows significantly.

Estimation of $CTE_{\alpha}(X, Y)$ on simulated data

Compare the estimated results with the theoretical ones (simple cases-explicit value of the theoretical $\mathrm{CTE}_{\alpha}(X, Y)$).

In the following we denote $\widehat{\mathrm{CTE}}_{\alpha}(X, Y)$ the mean of the two estimated coordinates of $\widehat{\mathrm{CTE}}_{\alpha}(X, Y)$ on 100 simulations. F_n empirical estimator of the bivariate distribution function, with $n = 1000$.

α	$\mathrm{CTE}_{\alpha}(X, Y)$	$\widehat{\mathrm{CTE}}_{\alpha}(X, Y)$	$\widehat{\sigma}$	MSE
0.10	(0.627, 0.627)	(0.571, 0.641)	(0.021, 0.021)	(0.062, 0.024)
0.24	(0.761, 0.761)	(0.703, 0.785)	(0.025, 0.026)	(0.068, 0.035)
0.38	(0.896, 0.896)	(0.838, 0.921)	(0.034, 0.034)	(0.071, 0.044)
0.52	(1.051, 1.051)	(0.995, 1.076)	(0.048, 0.049)	(0.077, 0.058)
0.66	(1.246, 1.246)	(1.184, 1.281)	(0.069, 0.074)	(0.093, 0.081)
0.80	(1.531, 1.531)	(1.472, 1.564)	(0.123, 0.121)	(0.141, 0.131)

Table: (X, Y) with independent and exponentially distributed components with parameter $\lambda = 2$.

Estimation of $CTE_\alpha(X, Y)$ on simulated data

α	$\mathrm{CTE}_\alpha(X, Y)$	$\widehat{\mathrm{CTE}}_\alpha(X, Y)$	$\widehat{\sigma}$	MSE
0.10	(1.255, 0.627)	(1.206, 0.649)	(0.039, 0.021)	(0.063, 0.031)
0.24	(1.521, 0.761)	(1.475, 0.783)	(0.048, 0.027)	(0.067, 0.034)
0.38	(1.792, 0.896)	(1.743, 0.917)	(0.067, 0.035)	(0.082, 0.041)
0.52	(2.102, 1.051)	(2.051, 1.072)	(0.094, 0.043)	(0.107, 0.049)
0.66	(2.492, 1.246)	(2.426, 1.274)	(0.141, 0.075)	(0.153, 0.081)
0.80	(3.061, 1.531)	(3.016, 1.554)	(0.312, 0.132)	(0.323, 0.133)

Table: (X, Y) with independent and exponentially distributed components with parameter $\lambda = 1$ and $\lambda = 2$ respectively.

Estimation of $CTE_\alpha(X, Y)$ on simulated data

We estimate the $\mathrm{CTE}_\alpha(X, Y)$ for $\alpha = 0.9$ and for increasing values of the sample size n. The theoretical value is $\mathrm{CTE}_{0.9}(X, Y) = (3.78, 1.89)$.

n	500	1000	1500	2000	2500	5000	10000
MSE	(0.76, 0.37)	(0.65, 0.2)	(0.53, 0.2)	(0.56, 0.12)	(0.51, 0.12)	(0.48, 0.1)	(0.24, 0.01)

Table: Evolution of the mean square error in terms of the size of sample n for $\alpha = 0.9$; (X, Y) independent and exponentially distributed components with parameter $\lambda = 1$ and $\lambda = 2$ respectively.

As expected we need a larger sample size n to get the same performances as for lower levels. This phenomenon can be explained by the magnitude of the constant A but also by the deteriorated performance of the empirical estimator F_n for hight value of α.

Estimation of $CTE_\alpha(X, Y)$ on real data

We consider **Loss-ALAE data** (for details see Frees and Valdez, 1998). Each claim consists of an indemnity payment (the loss, X) and an allocated loss adjustment expense (ALAE, Y).

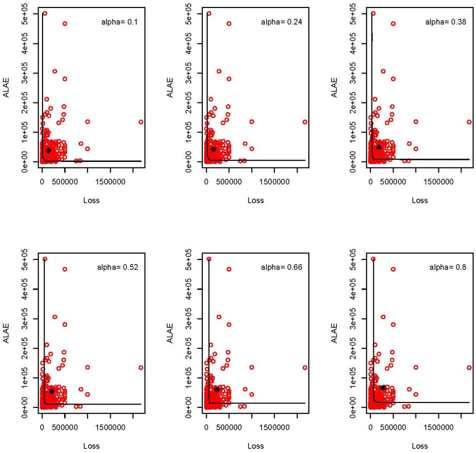
α	0.10	0.24	0.38
$\widehat{\mathrm{CTE}}_\alpha(X, Y)$	(143228.5, 38671.3)	(159738.8, 42482.3)	(186690.5, 49075.1)
α	0.52	0.66	0.80
$\widehat{\mathrm{CTE}}_\alpha(X, Y)$	(212155.7, 53742.1)	(234374.8, 60090.1)	(275828.1, 66735.1)

Table: $\widehat{\mathrm{CTE}}_\alpha(X, Y)$ for Loss-ALAE data, with different values of α.

In this real setting the estimation of CTE_α can be used in order to quantify the mean of the Loss (resp. ALAE) conditionally to the fact that Loss and ALAE data belong jointly to the specific risk's area $L_{(X,Y)}(\alpha)$.

Estimation of $CTE_\alpha(X, Y)$ on real data

Loss-ALAE data: $\mathrm{VaR}_\alpha(F_n)$ (line) and $\widehat{\mathrm{CTE}}_\alpha(X, Y)$ (star).



Contents

Conclusions and perspectives

We have provided:

- Convergence results for the plug-in estimator of the levels sets of an unknown distribution function (in terms of d_H and d_λ).
- Possible applications in multivariate risk theory (VaR and CTE).
- Illustrations on simulated and real data sets.

Further developments:

- Rate of convergence for d_λ.
- Behavior of our estimator for high values of the level (using suitable F_n - Extreme Value Theory).
- Analysis of the CTE as risk measure: study of the classical properties (monotonicity, translation invariance, positive homogeneity, ...), relation between our CTE and the dependence structure (definition of some statistic order; orthant order of the components, ...)

Thank for your attention.

- A. Baíllo, J.A. Cuesta-Albertos, A. Cuevas, *Convergence rates in nonparametric estimation of level sets*. Statistics and Probability Letters 53, (2001), 27-35.
- F. Belzunce, A. Castaño, A. Olvera-Cervantes, A. Suárez-Llorens, *Quantile curves and dependence structure for bivariate distributions*. Computational Statistics & Data Analysis 51, (2007), 5112-5129.
- G. Biau, B. Cadre, B. Pelletier, *A graph-based estimator of the number of clusters*. ESAIM, Probability and Statistics 11, (2007), 272-280.
- J. Cai, H. Li, *Conditional tail expectations for multivariate phase-type distributions*. Journal of Applied Probability 42(3), (2005), 810-825.
- L. Cavalier, *Nonparametric estimation of regression level sets*. Statistics 29, (1997), 131-160.
- A. Cuevas, R. Fraiman, *A plug-in approach to support estimation*. Annals of Statistics 25, (1997), 2300-2312.
- A. Cuevas, W. González-Manteiga, A. Rodríguez-Casal, *Plug-in estimation of general level sets*. Australian and New Zealand Journal of Statistics 48, (2006), 7-19.

- A. Cuevas, A. Rodríguez-Casal, *On boundary estimation*. Advances in Applied Probability 36, (2004), 340-354.
- M. Denuit, J. Dhaene, M. Goovaerts, R. Kaas. *Actuarial Theory for Dependent Risks*. Wiley, (2005).
- P. Embrechts, G. Puccetti, *Bounds for functions of multivariate risks*. Journal of Multivariate Analysis 97(2), (2006), 526-547.
- J.R. Fernéandez-Ponce, A. Suárez-Lloréns, *Central Regions for Bivariate Distributions*. Austrian Journal of Statistic 31(2-3), (2002), 141-156.
- E.W. Frees, E.A. Valdez, *Understanding relationships using copulas*. North American Actuarial Journal 2, (1998), 1-15.
- T. Laloë, *Sur Quelques Problèmes d'Apprentissage Supervisé et Non Supervisé.*, Ph.D. Thesis, University of Montpellier II, (2009).
- G. Nappo, F. Spizzichino, *Relations between Kendall Distributions and level sets in bivariate exchangeable survival models*. Information Sciences. 179 (17), (2009), 2878-2890.

- W. Polonik, *Measuring mass concentration and estimating density contour clusters - an excess mass approach*. Annals of Statistics 23, (1995), 855-881.
- A. Rodríguez-Casal, *Estimacíon de conjuntos y sus fronteras. Un enfoque geometrico*. Ph.D. Thesis, University of Santiago de Compostela, (2003).
- C. Rossi, *On surface level curves of distribution in two variables*. Giornale dell'Istituto Italiano degli Attuari, Anno XXXVI- N.1-2, (1973), 87-108.
- L. Tibiletti, *Connectedness, arcwise-connectedness and convexity for levels sets of multidimensional distribution functions*. Ratio Mathematica 8, (1994), 21-27.
- L. Tibiletti, *Quasi-concavity property of multivariate distribution functions*. Ratio Mathematica 9, (1995), 27-36.
- L. Tibiletti, *On a new notion of multidimensional quantile*. Metron, vol. LI n.3-4, (1993), 77-83.
- L. Tibiletti, *On quasi-concavity of n-dimensional distribution functions*. Atti del XV convegno A.M.A.S.E.S., (1991), 503-515.
- A.B. Tsybakov, *On nonparametric estimation of density level sets*. Annals of Statistics 25, (1997), 948-969.