Asymptotic multivariate finite-time ruin probabilities with heavy-tailed claim amounts: Impact of dependence and optimal reserve allocation

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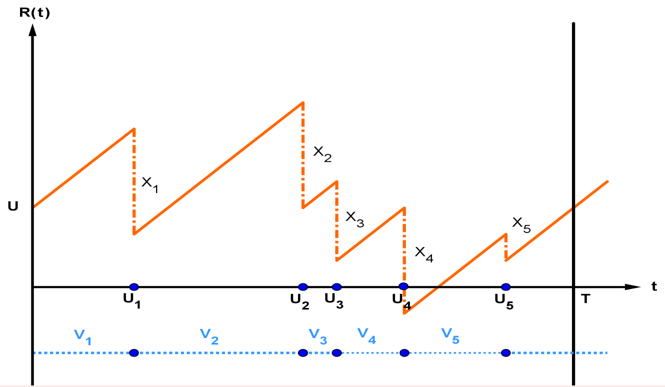
Outline

1. Introduction
   - Univariate ruin theory
   - Multivariate ruin theory
2. Asymptotics of multivariate finite-time ruin probabilities
3. Optimal allocation

Univariate Risk process : Insurance company's reserve evolution

$$R(t) = u + ct - \sum_{i=1}^{N(t)} X_i \; .$$



Classical assumptions

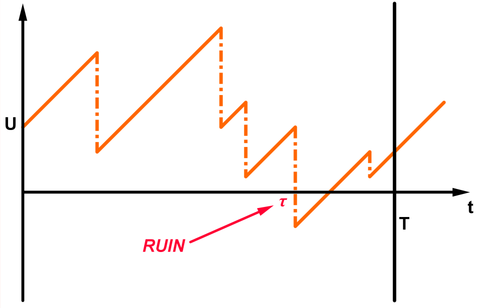
$$R(t) = u + ct - \sum_{i=1}^{N(t)} X_i ,$$

where

- $(N(t))_{t\geq 0}$: Poisson process with parameter λ.
- Claim amounts $(X_i)_{i\geq 1}$: sequence of independent and identically distributed positive random variables.
- Claim inter-occurrence times $(V_i)_{i\geq 1}$: sequence of independent and identically distributed random variables.
- $(X_i)_{i\geq 1}$ is independent from $(V_i)_{i\geq 1}$.

Remark : by convention, $\sum_{i=1}^{N(t)} X_i = 0$ if $N(t) = 0$.

Classical problems



- Finite-time ruin probability:

$$\psi(u, T) = P(\exists \tau \in [0, T]\ ,\ R(\tau) < 0 | R(0) = u),$$

- and infinite-time ruin probability:

$$\psi(u) = \lim_{T \to \infty} \psi(u, T).$$

Motivations

- Insurance company with multiple lines of business
- Univariate setting $\Rightarrow$ Multivariate setting
- Dependence between the lines of business
- Multivariate ruin probabilities
- Optimal initial reserve allocation

Multivariate risk process

$$\mathbf{R}(t) = \mathbf{a}u + \mathbf{c}t - \sum_{i=1}^{N(t)} \mathbf{X}_i \ , \tag{1}$$

where

- $\mathbf{a} = (a^{(1)}, \ldots, a^{(d)}) \in (0,1)^d$, $\mathbf{c} = (c^{(1)}, \ldots, c^{(d)}) \in \mathbb{R}_+^d$,
- $(N(t))_{t \geq 0}$ is a Poisson process with parameter $\lambda > 0$,
- $(\mathbf{X}_i)_{i \geq 1} = (X_i^{(1)}, \ldots, X_i^{(d)})_{i \geq 1}$ is a sequence of i.i.d. random vectors, independent from $N(t)$.

We denote, for $j = 1, \ldots, d$,

- $u^{(j)} = a^{(j)} u$,
- $S^{(j)}(t) = \sum_{i=1}^{N(t)} X_i^{(j)}$, and
- $Y^{(j)}(t) = S^{(j)}(t) - c^{(j)} t$.

Remark : by convention, $\sum_{i=1}^{N(t)} \mathbf{X_i} = 0$ if $N(t) = 0$.

Multivariate ruin probabilities

Let $T > 0$ and $\mathbf{u} = (u^{(1)}, \ldots, u^{(d)}) \in (0, \infty)^d$. Define the probability that

The sum of the line reserves becomes negative before T

$$\psi_{sum}(\mathbf{u}, T) = P\left(\sup_{[0,T]} \left\{ \sum_{j=1}^{d} Y^{(j)}(t) \right\} > \sum_{j=1}^{d} u^{(j)} \right) ; \qquad (2)$$

All the line reserves become negative before T

$$\psi_{and}(\mathbf{u}, T) = P\left(\bigcap_{j=1}^{d} \left\{ \sup_{[0,T]} Y^{(j)}(t) > u^{(j)} \right\} \right) ; \qquad (3)$$

One of the line reserve becomes negative before T

$$\psi_{or}(\mathbf{u}, T) = P\left(\bigcup_{j=1}^{d} \left\{ \sup_{[0,T]} Y^{(j)}(t) > u^{(j)} \right\} \right) ; \qquad (4)$$

All the line reserves are negative at a given time before T

$$\psi_{sim}(\mathbf{u}, T) = P\left(\sup_{[0,T]} \left(min\{ Y^{(1)}(t) - u^{(1)}, \ldots, Y^{(d)}(t) - u^{(d)} \} \right) > 0 \right) . \qquad (5)$$

With transfer capital allowed

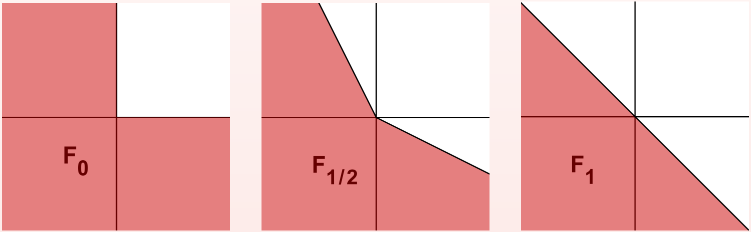
For $\beta \in [0,1]$, define

$$F_\beta = \left\{ \mathbf{x} : \beta \sum_{k=1}^{d} \left(x^{(k)} \vee 0 \right) < - \sum_{k=1}^{d} \left(x^{(k)} \wedge 0 \right) \right\} ,$$

where $\vee = \min$ and $\wedge = \max$.
For $T > 0$, define

$$\psi_{d,\beta}(u, T) = P\left(\exists t \in [0, T], \;\; \mathbf{R}(t) \in F_\beta \right) .$$

$\Longrightarrow$ *Transfer of fraction β of positive line allowed.*



1. Introduction

2. Asymptotics of multivariate finite-time ruin probabilities
    - Regular variation framework
    - A simple model of dependence
    - A Poisson shock model

3. Optimal allocation

Framework : Regular variation distributions

Univariate setting	Multivariate setting
A random variable X is regularly varying if there exists some $\alpha > 0$ such that, for all $t > 0$, $$P(X > tx) \underset{x\to\infty}{\sim} t^{-\alpha} P(X > x).$$ We denote $X \in \mathcal{R}_{-\alpha}$.	A random vector $\mathbf{X}$ is regularly varying if there exists a non-null Radon measure μ such that, for all Borel sets A bounded away from $\mathbf{0}$ with $\mu(\partial A) = 0$, $$P(\mathbf{X} \in xA) \underset{x\to\infty}{\sim} \mu(A) P(\lvert\mathbf{X}\rvert > x) .$$ We have for all $x > 0$ and A defined as above, $$\mu(xA) = x^{-\alpha}\mu(A) .$$ We denote $\mathbf{X} \in \mathcal{MR}_{\mu,-\alpha}$.

- Heavy-tailed distribution.
- Pareto-type distribution : $P(X > x) \propto x^{-\alpha}$, for some $\alpha > 0$.
- The smaller the α, the heavier the tail.
- Extreme events : earthquake, flooding, terrorist attacks...

Multivariate ruin probability [Hult and Lindskog, 2006]

For a risk process $(\mathbf{R}_t)_{t\geq 0}$ given by (1) with a common distribution $\mathbf{X} \in \mathcal{MR}_{-\alpha,\mu}$ for some $\alpha > 1$ and measure μ, we have for large u :

(**Infinite-time ruin probability**)

$$\psi_{\mathbf{d},\beta}(u,\infty) \sim \left(\int_0^\infty \mu(v\mathbf{p} + \mathbf{a} - F_\beta)dv\right) uP(|\mathbf{X}| > u) ,$$

with $\mathbf{p} = \lambda\mathbf{c} - E(\mathbf{X})$,

and (**Finite-time ruin probability**)

$$\psi_{\mathbf{d},\beta}(u, T) \sim (\lambda T)\mu(\mathbf{a} - F_\beta)P\left(|\mathbf{X}| > u\right).$$

Dependence structure in $\mathbf{X} \Rightarrow \mu$.

Simple forms for $\mathbf{X}$

Let X be a positive random variable which is regularly varying with some $\alpha > 1$.

- **Case 1)** If for some $1 \leq j \leq d$, $\mathbf{X} = X\mathbf{e}_j$, then $\mathbf{X} \in \mathcal{MR}_{-\alpha,\mu}$ for some measure μ and we have

$$\mu(\mathbf{a} - F_\beta) = \left(\beta + a^{(j)}(1-\beta)\right)^{-\alpha} .$$

- **Case 2)** If, for some $1 \leq k \leq d$,$\mathbf{X} = X\left(\sum_{i=1}^k \mathbf{e}_k\right) = X\mathbf{1}_k$, then $\mathbf{X} \in \mathcal{MR}_{-\alpha,\mu}$ for some measure μ and we have

$$\mu(d^{-1}\mathbf{1} - F_\beta) = \left(d^{-1}\left(\frac{\beta(d-k)}{k} + 1\right)|\mathbf{1}_k|\right)^{-\alpha} .$$

- **Case 3)** If $\mathbf{X} = X\mathbf{1}$, then $\mathbf{X} \in \mathcal{MR}_{-\alpha,\mu}$ for some measure μ and we have

$$\mu(\mathbf{a} - F_\beta) = \left(\frac{\sum_{i=1}^{k^*} a^{(i:d)} + \beta\sum_{i=k^*+1}^{d} a^{(i:d)}}{k^* + \beta(d-k^*)}|\mathbf{1}|\right)^{-\alpha} ,$$

where for $1 \leq i \leq d$, $a^{(i:d)}$ is the ith larger component of $\mathbf{a}$ and

$$k^* = \inf\left\{k \in [1, d-1] : a^{(k+1:d)} > \frac{\sum_{i=1}^{k} a^{(i:d)} + \beta\sum_{i=k+1}^{d} a^{(i:d)}}{k + \beta(d-k)}\right\} .$$

A simple model of dependence

$\mathbf{X} = (X^{(1)}, ..., X^{(d)})$ is such that, for $1 \leq j \leq d$,

$$X^{(j)} = I^{(j)} W^{(0)} + (1 - I^{(j)}) W^{(j)},$$

where,

- $(W^{(j)})_{0 \leq j \leq d}$ is an i.i.d. non-negative random vector with common distribution $W \in \mathcal{R}_{-\alpha}$, for some $\alpha > 1$, and cdf F,
- and $(I^{(j)})_{1 \leq j \leq d}$ is a vector of independent Bernoulli random variables with same parameter $p \in [0, 1]$, and independent from $(W^{(j)})_{0 \leq j \leq d}$.

Assume $\mathbf{a} = d^{-1}\mathbf{1}$. We have, for $T > 0$ and large u,

$$\psi_{d,\beta}(u, T) \sim \Big\{ (1-p)^d d \left((d-1)\beta + 1\right)^{-\alpha} + \sum_{k=1}^{d} \binom{d}{k} p^k (1-p)^{d-k} \left[\left(\left(\frac{d-k}{k} \right) \beta + 1 \right)^{-\alpha} + (d-k)\left((d-1)\beta + 1\right)^{-\alpha} \right] \Big\} \times d^{\alpha} (\lambda T) \overline{F}(u).$$

A Poisson shock model (1)

Let for $1 \leq j \leq d$,

$$S^{(j)}(t) = \sum_{k=1}^{N_0(t)} X_k^{(0)} + \sum_{k=1}^{N_j(t)} X_k^{(j)} ,$$

where

- All the r.v.'s are mutually independent.
- $(N_0(t))_{t\geq 0}$, $(N_j(t))_{t\geq 0}$ $1 \leq j \leq d$ are Poisson processes respectively with parameter λ_0, λ_j $1 \leq j \leq d$.
- $X_k^{(j)} \sim X \in \mathcal{R}_{-\alpha}$ (cdf F, $\alpha > 1$) for $0 \leq j \leq d$ and $k \geq 1$.

$$\Rightarrow \mathbf{S}_t = \sum_{k=1}^{N_0(t)} X_k^{(0)} \mathbf{1} + \sum_{j=1}^{d} \sum_{k=1}^{N_j(t)} X_k^{(j)} \mathbf{e}_j = \sum_{k=1}^{N(t)} \mathbf{X}_k ,$$

where

- $N(t) = \sum_{j=0}^{d} N_j(t)$ is a Poisson process with parameter $\overline{\lambda} = \lambda^{(0)} + \lambda^{(1)} + \ldots + \lambda^{(d)}$, and
- $\mathbf{X}_k = X_k^{(0)} \mathbf{1} \delta_0(\xi_k) + \sum_{j=1}^{d} X_k^{(j)} \mathbf{e}_j \delta_j(\xi_k)$ with, $(\xi_k)_{k\geq 1}$ an i.i.d. sequence of r.v. with $P(\xi_k = j) = \lambda^{(j)}/\overline{\lambda}$ for $k \geq 1$ and $0 \leq j \leq d$.

A Poisson shock model (2)

Under the above assumptions, we have, for $T > 0$ and large u,

$$\psi_{d,\beta}(u,T) \sim \left\{ \lambda^{(0)} \left[\frac{\sum_{j=1}^{k^*} a^{(j:d)} + \beta \sum_{j=k^*+1}^{d} a^{(j:d)}}{k^* + \beta(d-k^*)} \right]^{-\alpha} + \sum_{j=1}^{d} \lambda^{(j)} \left[\beta + a^{(j)}(1-\beta) \right]^{-\alpha} \right\} T\overline{F}(u) ,$$

where for $1 \leq j \leq d$, $a^{(j:d)}$ is jth larger component of **a** and

$$k^* = \inf \left\{ k \in [1, d-1] : a^{(k+1:d)} > \frac{\sum_{j=1}^{k} a^{(j:d)} + \beta \sum_{j=k+1}^{d} a^{(j:d)}}{k + \beta(d-k)} \right\} .$$

A Poisson shock model (3)

When no transfer is allowed ($\beta = 0$), we get, for large u and $T > 0$,

$$\psi_{d,0}(u,T) \sim \left\{ \lambda^{(0)} \left[\min_{1\le j\le d} \{a^{(j)}\} \right]^{-\alpha} + \sum_{j=1}^{d} \lambda^{(j)} \left[a^{(j)} \right]^{-\alpha} \right\} T\overline{F}(u) .$$

This result corresponds to ψ_{or} (4).

When transfer is allowed without restriction ($\beta = 1$), we get, for large u and $T > 0$,

$$\psi_{d,1}(u,T) \sim \left\{ \lambda^{(0)}(d^{\alpha} - 1) + 1 \right\} T\overline{F}(u) .$$

This result corresponds to ψ_{sum} (2).

Optimal Problem

Here, we are going to minimize the asymptotics of $\psi_{d,\beta=0}(u,T)$ we denote by $\tilde{\psi}_d(u,T)$.

$$\begin{cases} \min\limits_{\mathbf{a}\in(0,1)^d} \tilde{\psi}_d(u,T) \, , \\ \text{under the constraint } a^{(1)} + \ldots + a^{(d)} = 1 \, . \end{cases} \tag{6}$$

- **Case 1 :** the company is composed of d lines of business and they are mutually independent.
- **Case 2 :** the company is composed of two lines of business and their dependence structure is described by the Poisson shock model.
- **Case 3 :** the company is composed of three lines of business, one is independent from the others and the two others are dependent via the Poisson shock model.

Case 1 : Independence

In this case, we have for $T > 0$ and large u,

$$\psi_d(u,T) \sim \left\{ \sum_{j=1}^{d} \lambda^{(j)} \left[a^{(j)} \right]^{-\alpha} \right\} T\overline{F}(u) \,.$$

and the solution of (6) is, for all $1 \leq i \leq d$,

$$a^{(i)^*} = \left\{ \frac{\lambda^{(i)\,\frac{1}{\alpha+1}}}{\sum_{j=1}^{d} \lambda^{(j)\,\frac{1}{\alpha+1}}} \right\} \,.$$

Case 2 : Two dependent lines

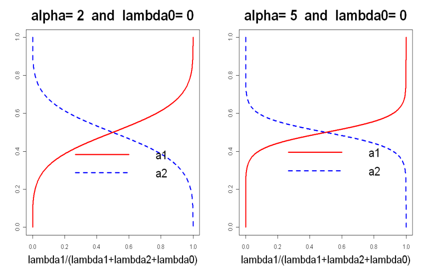
In this case, we have for $T > 0$ and large u,, with $a = a^{(1)}$

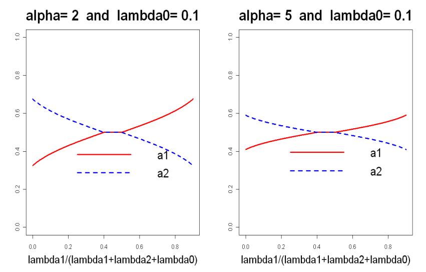
$$\psi_2(u,T) \sim \left\{ \lambda^{(0)} \left[\min(a;1-a)\right]^{-\alpha} + \lambda^{(1)} a^{-\alpha} + \lambda^{(2)} (1-a)^{-\alpha} \right\} T\overline{F}(u)\,.$$

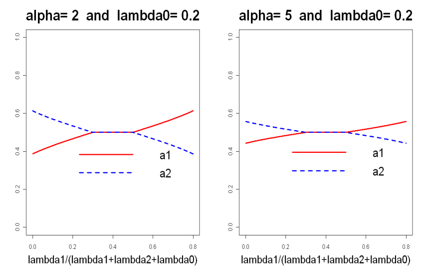
and the solution of (6) is

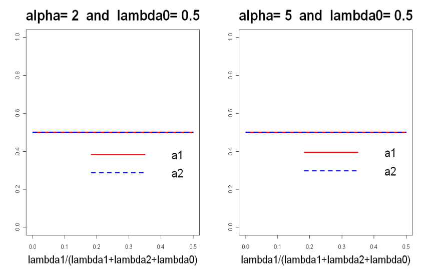
$$a* = \begin{cases} \frac{1}{2} & \text{if } \lambda^{(0)} > |\lambda^{(1)} - \lambda^{(2)}|\,, \\ \frac{\lambda^{(1)\frac{1}{\alpha+1}}}{\lambda^{(1)\frac{1}{\alpha+1}} + \left(\lambda^{(0)}+\lambda^{(2)}\right)^{\frac{1}{\alpha+1}}} & \text{if } \lambda^{(0)} \le \lambda^{(1)} - \lambda^{(2)}\,, \\ \frac{\left(\lambda^{(0)}+\lambda^{(1)}\right)^{\frac{1}{\alpha+1}}}{\left(\lambda^{(0)}+\lambda^{(1)}\right)^{\frac{1}{\alpha+1}} + \lambda^{(2)\frac{1}{\alpha+1}}} & \text{if } \lambda^{(0)} \le \lambda^{(2)} - \lambda^{(1)}\,. \end{cases}$$

Case 2 : Two dependent lines









Case 3 : Two dependence lines, one independent

In this case, we have, for $T > 0$ and large u,

$$\psi_{\tilde{3}}(u,T) \sim \left\{ \lambda^{(0)} \left[\min(a^{(1)}; a^{(2)}) \right]^{-\alpha} + \lambda^{(1)} a^{(1)\,-\alpha} + \lambda^{(2)} a^{(2)\,-\alpha} + \lambda^{(3)} a^{(3)\,-\alpha} \right\} T\overline{F}(u)\,.$$

In this case, the solution of (6) is as follows.

- If $\lambda^{(0)} > |\lambda^{(1)} - \lambda^{(2)}|$, then

$$\begin{cases} a^{(1)*} = a^{(2)*} = \frac{1}{2} \dfrac{\left(2^{\alpha}\left(\lambda^{(0)}+\lambda^{(1)}+\lambda^{(2)}\right)\right)^{\frac{1}{\alpha+1}}}{\left(2^{\alpha}\left(\lambda^{(0)}+\lambda^{(1)}+\lambda^{(2)}\right)\right)^{\frac{1}{\alpha+1}} + \lambda^{(3)\,\frac{1}{\alpha+1}}}\,, \\ a^{(3)*} = \dfrac{\lambda^{(3)\,\frac{1}{\alpha+1}}}{\left(2^{\alpha}\left(\lambda^{(0)}+\lambda^{(1)}+\lambda^{(2)}\right)\right)^{\frac{1}{\alpha+1}} + \lambda^{(3)\,\frac{1}{\alpha+1}}}\,. \end{cases}$$

Case 3 : Two dependence lines, one independent

- If $\lambda^{(0)} \leq \lambda^{(1)} - \lambda^{(2)}$, then

$$\begin{cases} a^{(1)*} = \dfrac{\lambda^{(1)\,\frac{1}{\alpha+1}}}{\lambda^{(1)\,\frac{1}{\alpha+1}} + \left(\lambda^{(0)}+\lambda^{(2)}\right)^{\frac{1}{\alpha+1}} + \lambda^{(3)\,\frac{1}{\alpha+1}}}, \\ a^{(2)*} = \dfrac{\left(\lambda^{(0)}+\lambda^{(2)}\right)^{\frac{1}{\alpha+1}}}{\lambda^{(1)\,\frac{1}{\alpha+1}} + \left(\lambda^{(0)}+\lambda^{(2)}\right)^{\frac{1}{\alpha+1}} + \lambda^{(3)\,\frac{1}{\alpha+1}}}, \\ a^{(3)*} = \dfrac{\lambda^{(3)\,\frac{1}{\alpha+1}}}{\lambda^{(1)\,\frac{1}{\alpha+1}} + \left(\lambda^{(0)}+\lambda^{(2)}\right)^{\frac{1}{\alpha+1}} + \lambda^{(3)\,\frac{1}{\alpha+1}}}. \end{cases}$$

- If $\lambda^{(0)} \leq \lambda^{(2)} - \lambda^{(1)}$, then

$$\begin{cases} a^{(1)*} = \dfrac{\left(\lambda^{(0)}+\lambda^{(1)}\right)^{\frac{1}{\alpha+1}}}{\left(\lambda^{(0)}+\lambda^{(1)}\right)^{\frac{1}{\alpha+1}} + \lambda^{(2)\,\frac{1}{\alpha+1}} + \lambda^{(3)\,\frac{1}{\alpha+1}}}, \\ a^{(2)*} = \dfrac{\lambda^{(2)\,\frac{1}{\alpha+1}}}{\left(\lambda^{(0)}+\lambda^{(1)}\right)^{\frac{1}{\alpha+1}} + \lambda^{(2)\,\frac{1}{\alpha+1}} + \lambda^{(3)\,\frac{1}{\alpha+1}}}, \\ a^{(3)*} = \dfrac{\lambda^{(3)\,\frac{1}{\alpha+1}}}{\left(\lambda^{(0)}+\lambda^{(1)}\right)^{\frac{1}{\alpha+1}} + \lambda^{(2)\,\frac{1}{\alpha+1}} + \lambda^{(3)\,\frac{1}{\alpha+1}}}. \end{cases}$$

Thank you for your attention !