
On Some Riemannian Aspects of Two and Three-Body Controlled Problems*

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Dedicated to Joseph Noailles

Summary. The flow of the Kepler problem (motion of two mutually attracting bodies) is known to be geodesic after the work of Moser [21], extended by Belbruno and Osipov [2, 22]: Trajectories are reparameterizations of minimum length curves for some Riemannian metric. This is not true anymore in the case of the three-body problem, and there are topological obstructions as observed by McCord *et al.* [20]. The controlled formulations of these two problems are considered so as to model the motion of a spacecraft within the influence of one or two planets. The averaged flow of the (energy minimum) controlled Kepler problem with two controls is shown to remain geodesic. The same holds true in the case of only one control provided one allows singularities in the metric. Some numerical insight into the control of the circular restricted three-body problem is also given.

Key words: Two and three-body problems, geodesic flow, optimal control, cut and conjugate loci

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1 Introduction

The circular restricted three-body problem is defined as follows [26].

Two bodies describe circular orbits around their center of mass under the influence of their mutual gravitational attraction, and a third one (attracted by the previous two but not influencing their motion) evolves

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in the plane defined by the two rotating ones. The restricted problem is to describe the motion of this third body.

We investigate the optimal control of this problem. The two primaries are planets, typically Earth and Moon, the third body is a spacecraft. The control is the thrust of this spacecraft. A recent example of this problem is the SMART-1 mission [4, 23] of the European Space Agency in the Earth-Moon system. This case has three important features: (i) Provided we neglect non-coplanar effects, the circular restricted model is germane to the problem as the eccentricity of the Moon orbit is about 0.0549; (ii) The mass m_2 of the second primary (the Moon) is much smaller than the mass of the first (the Earth), m_1 , so that $\mu = m_2/(m_1 + m_2) \simeq 0.0121$ is a small parameter of the model; (iii) The thrust of the engine is very low since solar-electric propulsion is used (around 0.07 Newtons for a 350 Kilogram vehicle), so the magnitude of the control is another small parameter.

In a rotating frame, the dynamics is normalized to the second order mechanical system

$$\ddot{q} + \nabla V_\mu(q) + 2iq = \varepsilon u, \quad |u| = \sqrt{u_1^2 + u_2^2} \leq 1.$$

Coordinate $q \in \mathbf{C} \simeq \mathbf{R}^2$ is the position vector while u is the control (the normalized acceleration, here). In this moving frame, the circular restricted three-body potential is

$$V_\mu(q) = -\frac{q^2}{2} - \frac{1-\mu}{r_1} - \frac{\mu}{r_2},$$

$$r_1^2 = (q_1 + \mu)^2 + q_2^2, \quad r_2^2 = (q_1 - 1 + \mu)^2 + q_2^2.$$

Parameter μ is the ratio $m_2/(m_1 + m_2)$ of the masses of the two primaries, and ε is the bound on the acceleration. When μ vanishes, we have a controlled two-body problem. The uncontrolled equations of motion can also be written in Hamiltonian form using *Jacobi* first integral (total energy),

$$J_\mu(q, \dot{q}) = \frac{|\dot{q}|^2}{2} + V_\mu(q).$$

In complex notation, let $p = \dot{q} + iq$. Then

$$-J_\mu(q, p) = \frac{|p|^2}{2} + p_2q_1 - p_1q_2 + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}.$$

The controlled system with Hamiltonian drift is so

$$\dot{q} = \frac{\partial J_\mu}{\partial p}, \quad \dot{p} = -\frac{\partial J_\mu}{\partial q} + \varepsilon u, \quad |u| \leq 1. \tag{1}$$

In the case of two bodies ($\mu = 0$) and no control ($\varepsilon = 0$), the equations of motion in a fixed frame are

$$\ddot{q} + \frac{q}{|q|^3} = 0, \quad q \in \mathbf{R}^2 - \{0\}. \tag{2}$$

In Hamiltonian form,

$$\dot{q} = \frac{\partial J_0}{\partial p}, \quad \dot{p} = -\frac{\partial J_0}{\partial q},$$

with energy $J_0 = |\dot{q}|^2/2 - 1/|q| = |p|^2/2 - 1/|q|$, as $p = \dot{q}$ in the fixed frame. It was proven in [21] that, for negative values of the energy, the Hamiltonian flow of the system is a reparameterization of the geodesic flow on the punctured two-sphere, $\hat{\mathbf{S}}^2 = \mathbf{S}^2 - \{N\}$ (North pole removed). We sketch the construction in dimension $n \geq 2$ where the result holds unchanged. (Take $q \in \mathbf{R}^n - \{0\}$ in (2).) One first identifies the tangent bundle of the punctured n -sphere with the set of vectors $\xi = (\xi_0, \dots, \xi_n)$, $\eta = (\eta_0, \dots, \eta_n)$ of \mathbf{R}^{n+1} such that

$$|\xi| = 1, \quad (\xi|\eta) = 0.$$

The puncture is obtained by removing $\xi_0 = 1$. Then, the transformation from the tangent bundle to \mathbf{R}^{2n} is

$$q_i = (1 - \xi_0)\eta_i + \eta_0\xi_i, \quad p_i = -\frac{\xi_i}{1 - \xi_0}, \quad i = 1, \dots, n.$$

Provided time is changed according to

$$dt = |q|ds, \tag{3}$$

the Hamiltonian flow on $J_0 = -1/2$ is mapped into the Hamiltonian flow on $\tilde{J}_0 = 1/2 \subset T\hat{\mathbf{S}}^n$ where

$$\tilde{J}_0(\xi, \eta) = \frac{1}{2}|\xi|^2|\eta|^2.$$

This level set is the *unit* or *spherical* tangent bundle of $\hat{\mathbf{S}}^2$ since $|\eta| = 1$. There,

$$\xi' = \eta, \quad \eta' = -\xi,$$

so $\xi'' + \xi = 0$ and one actually gets geodesics parameterized by arc length. The Levi-Civita change in time (3) regularizes the collision and the dynamics is extended on the whole n -sphere. The result of [21] was further generalized to cover the case of zero or positive energy levels by [2] and [22].

Trajectories in optimal control are projections of Hamiltonian flows, in general with singularities described by Pontryagin maximization condition. Riemannian problems being the simplest instance of control problems, one may ask whether a given smooth Hamiltonian flow is the reparameterization of some Riemannian flow as in the two-body case. This question is addressed in [20], noting the following fact. Given a flow on an odd $2n - 1$ -dimensional manifold M , a necessary condition for the flow to be geodesic is that the manifold be the unit tangent bundle of some other manifold of dimension n . This puts topological restrictions on M . These conditions are expressed in

terms of the homology of M and applied to the (general) three-body problem. We state the results and recall some basic facts on homology [17].

On a topological space X , a *singular p -simplex* is a continuous map $\sigma_p : \Delta_p \rightarrow X$. Here, Δ_p is the standard p -simplex, that is the set of $(t_0, \dots, t_p) \in \mathbf{R}^{p+1}$ such that

$$t_0 + \dots + t_p = 1, \quad t_i \geq 0, \quad i = 0, \dots, p.$$

Let v_0, \dots, v_p be its vertices. The set $C_p(X)$ of p -chains is the free abelian group generated by singular p -simplices. The *boundary operator* $\partial_p : C_p(X) \rightarrow C_{p-1}(X)$ is

$$\partial_p(\sigma_p) = \sum_{i=0}^p (-1)^i \sigma_p|_{\Delta(v_0, \dots, \hat{v}_i, \dots, v_p)}$$

where the restriction is on the $(p-1)$ -simplex $\Delta(v_0, \dots, \hat{v}_i, \dots, v_p)$ with vertex v_i removed, implicitly identified with Δ_{p-1} . Images of $(p+1)$ -chains by ∂_{p+1} are p -boundaries, and p -chains in the kernel of ∂_p are p -cycles. As $\delta_p \delta_{p+1} = 0$, boundaries are cycles while, conversely, one defines the p -th homology group $H_p(X)$ as the quotient

$$H_p(X) = \text{Ker } \partial_p / \text{Im } \partial_{p+1}.$$

The rank of the \mathbf{Z} -module $H_p(X)$ is β_p , the p -th Betti number, and the Euler-Poincaré characteristic of M is

$$\chi(M) = \sum_{p=0}^n \beta_p (-1)^p.$$

Proposition 1 ([20]). *If M is a non-compact connected orientable manifold of dimension $2n - 1$, a necessary condition for it to be the unit tangent bundle of some orientable n -manifold is $\beta_{n-1} \neq 0$.*

Applying this condition to the three-body problem, one gets the following negative result.

Theorem 1 ([20]). *The flow of the planar three-body problem on a negative level of energy is not geodesic.*

In the case of controlled two and three-body problems, there is not much hope to retrieve Riemannian flows, unless one uses some approximation process. The paper is organized as follows. In section 2, we examine the case of two bodies and two controls. Using averaging on a relaxed problem, we show that the flow is Riemannian when the L^2 -norm of the control is minimized. Its properties are essentially captured by those of a metric on the two-sphere. The same holds true for the case of two bodies and one control (directed by velocity) provided one allows singularities in the metric. This is addressed

in section 3. A preliminary discussion of the restricted three-body and two-control case is made in section 4. The problem is control-affine, with a drift. One can still define the exponential mapping associated with minimum time extremals and compute conjugate points to ensure, as in the Riemannian case, local optimality of trajectories.

2 Two bodies, two controls

We consider an L^2 -relaxation of the controlled two-body problem. The bound ε on the control is dropped,

$$\ddot{q} + \frac{q}{|q|^3} = u, \quad u \in \mathbf{R}^2, \quad (4)$$

while the final time, t_f , is a fixed parameter of the criterion:

$$\int_0^{t_f} |u|^2 dt \rightarrow \min.$$

In the sub-Riemannian case, L^2 -minimization parameterized by final time can be recast as a minimum time problem with a bound on the control. Both problems coincide, so t_f and ε play dual roles in this sense. The situation is more complicated here because of the Kepler drift in the motion. In order to identify a new small parameter of the problem and perform averaging, we notice that the negative energy level $J_0 < 0$ has a trivial fiber structure. This is apparent in suited geometric coordinates.

The set X of oriented ellipses has moduli space the product manifold $\mathbf{R}_+^* \times \mathbf{S}^2$: Each ellipse is defined by its semi-major axis $a > 0$ (we exclude trivial orbits, $a = 0$), and to any point on \mathbf{S}^2 , (θ, φ) in standard spherical coordinates, is uniquely associated an eccentricity, $e = \sin \varphi$, an *argument of perigee* (angle of the semi-major axis with a fixed reference axis), θ , and an orientation. The orientation of the ellipse changes when the point goes from one hemisphere to the other. Collision orbits correspond to the equator $\varphi = \pi/2$ and are included in the model.

Remark 1. Ellipses associated with the poles or the equator have richer symmetries (automorphisms) than others. The moduli space is then said to be *coarse*. It remains finer that the moduli space of conformal ellipses where homothety and rotation moduli (a, θ) would be dropped.

Position on the orbit is defined by the polar angle in the plane or *longitude*, $l \in \mathbf{S}^1$. The state space is hence $\mathbf{S}^1 \times X$, and we have a trivial fiber space whose fiber is the moduli space. To each uncontrolled trajectory on $J_0 < 0$ corresponds a unique point in the fiber, so the drift in (4) has the form

$$F_0(l, x) = \omega(l, x) \frac{\partial}{\partial l}, \quad (l, x) \in \mathbf{S}^1 \times X.$$

(See (5) hereafter for the definition of ω .) Keeping the same notation, let then l be the *cumulated longitude*, associated with the covering

$$\mathbf{R} \ni l \mapsto e^{il} \in \mathbf{S}^1.$$

Choosing l as new time, we recast the problem as control-affine problem on X without drift but with non-autonomous vector fields depending periodically on l ,

$$\begin{aligned} \frac{dx}{dl} &= u_1 F_1(l, x) + u_2 F_2(l, x), \quad u \in \mathbf{R}^2, \\ \int_0^{l_f} |u|^2 \frac{dl}{\omega(l, x)} &\rightarrow \min \quad (\text{fixed } l_f). \end{aligned}$$

The two vector fields F_1, F_2 on X are periodic in the parameter l . Introducing *mean motion*, $n = a^{-3/2}$, and *true anomaly*, $\tau = l - \theta$, one gets

$$\begin{aligned} F_1(l, x) &= \frac{P^2}{W^2} \left(-\frac{3ne \sin \tau}{1 - e^2} \frac{\partial}{\partial n} + \sin \tau \frac{\partial}{\partial e} - \cos \tau \frac{1}{e} \frac{\partial}{\partial \theta} \right), \\ F_2(l, x) &= \frac{P^2}{W^2} \left(-\frac{3nW}{1 - e^2} \frac{\partial}{\partial n} + \left(\cos \tau + \frac{e + \cos \tau}{W} \right) \frac{\partial}{\partial e} \right. \\ &\quad \left. + \left(\sin \tau + \frac{\sin \tau}{W} \right) \frac{1}{e} \frac{\partial}{\partial \theta} \right), \end{aligned}$$

with $W = 1 + e \cos \tau$. The pulsation is

$$\omega(l, x) = \frac{nW^2}{(1 - e^2)^{3/2}}. \tag{5}$$

Averaging on the base space eliminates l , that is the drift in the equation.

The normal maximized Hamiltonian on $\mathbf{S}^1 \times T^*X$ is

$$H(l, x, p) = \frac{\omega}{2} (H_1^2 + H_2^2)(l, x, p),$$

where $H_i = \langle p, F_i(l, x) \rangle$, $i = 1, 2$, are the Hamiltonian lifts of the vector fields. Let

$$\bar{H}(x, p) = \frac{1}{2\pi} \int_0^{2\pi} H(l, x, p) dl \tag{6}$$

be the averaged Hamiltonian. As $1/l_f$, the new small parameter, tends to zero, the flow of \bar{H} converges uniformly towards the flow of H on $[0, l_f]$. (See [16].) It turns out that the averaged flow is the flow of some Riemannian metric on X , a result which can be traced back to Edelbaum [14]. We refer to [5, 9] for details.

Proposition 2. *The averaged Hamiltonian is*

$$H(x, p) = \frac{p_r^2}{2} + \frac{c^2}{2r^2} \left(\frac{1 - \lambda \sin^2 \varphi}{\sin^2 \varphi} p_\theta^2 + p_\varphi^2 \right)$$

with $r = (2/5)n^{5/6}$, $c = \sqrt{2/5}$ and $\lambda = 4/5$.

The metric is

$$dr^2 + \frac{r^2}{c^2} \left(\frac{\sin^2 \varphi}{1 - \lambda \sin^2 \varphi} d\theta^2 + d\varphi^2 \right).$$

It is Liouville integrable. The integration and the analysis of optimality can be made on the restriction to \mathbf{S}^2 by reparameterizing time according to $ds = c^2 dl/r^2$. (See [6].) This amounts to restricting to a coarser moduli space where homothetic ellipses are identified. The restricted metric is

$$XR(\lambda X)d\theta^2 + d\varphi^2 \quad \text{with} \quad R = \frac{1}{1 - X} \quad \text{and} \quad X = \sin^2 \varphi. \quad (7)$$

As $\chi(\mathbf{S}^2) = 2$, the two vector fields $(XR(X))^{-1/2} \partial\theta$, $\partial/\partial\varphi$ cannot form a global frame on the sphere. They have polar singularities that do not define genuine singularities of the metric.

Remark 2. Coordinates (θ, φ) are associated with the covering of the sphere with two punctures at North and South poles,

$$\mathbf{R} \times (0, \pi) \ni (\theta, \varphi) \mapsto (\sin \varphi \cos \theta, \sin \varphi \sin \theta, \cos \varphi) \in \mathbf{R}^3.$$

One retrieves the standard covering $\exp : \mathbf{C} \rightarrow \mathbf{C}^* \simeq \mathbf{S}^2 - \{N, S\}$ by putting $(\theta, \varphi) \mapsto \tan(\varphi/2) \exp(i\theta)$.

The Hamiltonian on \mathbf{S}^2 is $H_2 = (1/2)[(XR(\lambda X))^{-1} p_\theta^2 + p_\varphi^2]$. On the level $H_2 = 1/2$, time is arc length and we get the quadrature

$$Y^2 = 4(1 - X)[X - p_\theta^2(1 - \lambda X)], \quad Y = \dot{X}.$$

Since θ is cyclic, p_θ is constant (Clairaut first integral of a surface of revolution). The complex curve is of genus zero and admits a rational parameterization. We get

$$\sin z = \frac{1}{\delta^2 - p_\theta^2} [2\delta^2 X - (\delta^2 + p_\theta^2)], \quad dt = \frac{dz}{2\delta},$$

for $z \in \mathbf{R}$ and $\delta^2 = 1 + \lambda p_\theta^2$. We set $\theta_0 = 0$ by symmetry of revolution. We also assume $\varphi_0 = \pi/2$ without loss of generality since time translations generate any extremal on $H_2 = 1/2$ with arbitrary initial condition. The squared adjoint p_θ^2 is bounded by $1/(1 - \lambda)$.

Proposition 3. *The system for two bodies and two controls can be integrated using harmonic functions. One has*

$$\sin^2 \varphi = \frac{1}{2\delta^2} [(\delta^2 - p_\theta^2) \cos(2\delta t) + (\delta^2 + p_\theta^2)], \quad \delta^2 = 1 + \lambda p_\theta^2, \quad \lambda = 4/5,$$

$$\theta = \text{sign}(p_\theta) \left[\text{atan} \frac{(\delta^2 - p_\theta^2) + (\delta^2 + p_\theta^2) \tan(\delta t + \pi/4)}{2\delta p_\theta} \right]_0^t - \lambda p_\theta t.$$

Proof. The quadrature on θ is

$$\frac{d\theta}{dz} = \frac{p_\theta}{2\delta} \left(\frac{1}{X} - \lambda \right),$$

whence the result. □

Coordinate φ (*resp.* θ) is periodic (*resp.* quasi-periodic) with period $T = 2\pi/\delta = 2\pi/\sqrt{1 + \lambda p_\theta^2}$. (The period of φ is twice the period of $X = \sin^2 \varphi$.) The increment of θ over one period is important for the optimality analysis concluding the section. One has

$$\Delta\theta = 2\pi \left(1 - \frac{\lambda p_\theta}{\sqrt{1 + \lambda p_\theta^2}} \right). \tag{8}$$

Fix y_0 on \mathbf{S}^2 . The exponential mapping is defined for $t \in \mathbf{R}$ and $p_0 \in H_2(y_0, \cdot)^{-1}(1/2) \subset T_{y_0}^* \mathbf{S}^2$ by

$$\exp_{y_0} : (t, p_0) \mapsto \Pi \circ \exp t \vec{H}_2(y_0, p_0) = y(t, y_0, p_0)$$

where $\Pi : T^* \mathbf{S}^2 \rightarrow \mathbf{S}^2$ is the canonical projection and \vec{H}_2 the symplectic gradient. A *conjugate point* is a critical value of the exponential mapping. The time associated with such a critical point is the *conjugate time*, and one can define the *first conjugate point* along the geodesic associated with a given p_0 . The (first) *conjugate locus* is the set of all such points on geodesics emanating from y_0 . Jacobi theorem [13] asserts that, up to the first conjugate point, a geodesic is locally minimizing with respect to neighbouring continuous broken curves with same endpoints.

Theorem 2. *In the two-body two-control case, the conjugate locus of any point on the sphere has four (possibly degenerate) cusps, two horizontal and two meridional.*

Proof. According to [10] result, a sufficient condition is that $\Delta\theta$ is strictly decreasing convex. The condition is valid for (8). □

Finally, define the *cut time* along the geodesic defined by p_0 as the supremum of times t such that the geodesic $s \mapsto \exp_{y_0}(s, p_0)$ is globally minimizing on $[0, t]$. (See [13].) The corresponding point, if any, is the *cut point*. The *cut locus* is the set of all such points on geodesics emanating from y_0 . It is known since Poincaré that the cut locus of an analytic metric on the sphere is a finite tree whose extremities are singularities of the conjugate locus. In the case of a metric with more symmetries, the result can be specialized as follows.

Theorem 3 ([25]). *The cut locus of an analytic metric on the sphere of revolution with equatorial symmetry is an antipodal³ subarc provided the Gauss curvature is nondecreasing from North pole to equator.*

³ Symmetric with respect to the center of the sphere.

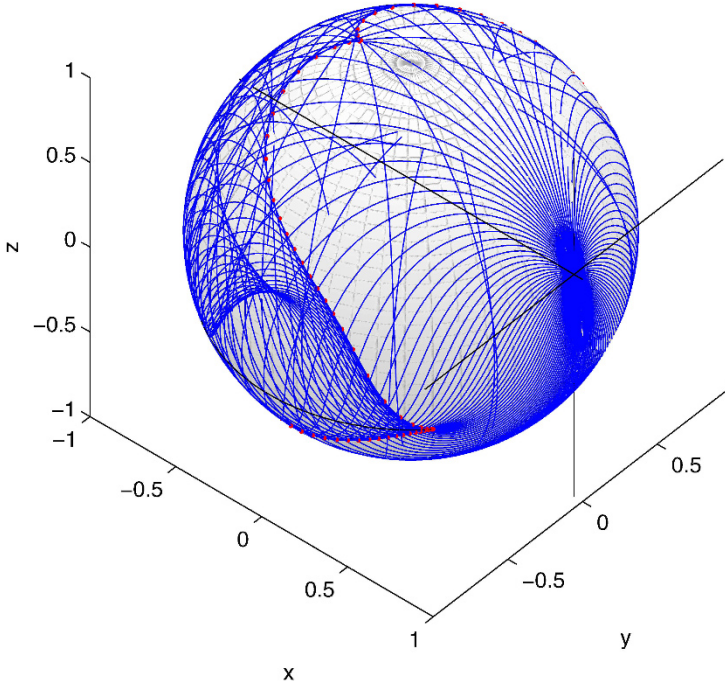


Fig. 1. Conjugate locus, two bodies and two controls. The astroid-shaped locus (in red) is the envelope of geodesics (in blue) emanating from the initial point. It has four (degenerate for initial condition on the poles) cusps, two horizontal and two meridional. The cut locus is a closed antipodal subarc (in black) whose extremities are horizontal cusps of the conjugate locus.

Though metric (7) has the required symmetries, the monotonicity condition on the curvature does not hold as

$$K = \frac{1 - \lambda(3 - 2X)}{(1 - \lambda X)^2}$$

is not decreasing when $X \in [0, 1]$ (remember that $X = \sin^2 \varphi$) for $\lambda = 4/5$. A refined result relying on $\Delta\theta$ being strictly decreasing still gives the result [10].

Theorem 4. *In the two-body two-control case, the cut locus of any point on the sphere is a closed antipodal subarc.*

Varying λ from zero to one in the definition of the metric (7), one connects the canonical metric on the sphere to a metric with an equatorial singularity,

$$\frac{\sin^2 \varphi}{1 - \sin^2 \varphi} d\theta^2 + d\varphi^2.$$

The original metric is conformal to the standard metric on an oblate ellipsoid of revolution with semi-minor axis $\sqrt{1 - \lambda}$ since

$$XR(\lambda X)d\theta^2 + d\varphi^2 = \frac{1}{1 - \lambda \sin^2 \varphi} [\sin^2 \varphi d\theta^2 + (1 - \lambda \sin^2 \varphi) d\varphi^2].$$

Making λ tend to one can be interpreted as letting the semi-minor axis tend to zero, thus collapsing the sphere on a two-face disk [7]. Such a singularity is intrinsic in the case of only one control as explained in next section.

3 Two bodies, one control

Consider the L^2 -minimization of the two-body problem with only one control acting tangentially [8],

$$\ddot{q} + \frac{q}{|q|^3} = u \frac{\dot{q}}{|\dot{q}|}, \quad u \in \mathbf{R}, \quad \int_0^{t_f} |u|^2 dt \rightarrow \min.$$

The state space is as before the trivial fiber space $\mathbf{S}^1 \times X$, $X = \mathbf{R}_+^* \times \mathbf{S}^2$, but we correct the relation between φ and the eccentricity,

$$e = \sin \varphi \sqrt{1 + \cos^2 \varphi}.$$

Changing again time to cumulated longitude,

$$\frac{dx}{dl} = u F_1(l, x), \quad \int_0^{l_f} |u|^2 \frac{dl}{\omega(l, x)} \rightarrow \min \quad (\text{fixed } l_f).$$

In (n, e, θ) coordinates,

$$F_1 = -\frac{3(1 - e^2)w}{n^{1/3}(1 + e \cos \tau)^2} \frac{\partial}{\partial n} + \frac{2(1 - e^2)^2}{n^{4/3}(1 + e \cos \tau)^2 w} \left[(e + \cos \tau) \frac{\partial}{\partial e} + \frac{\sin \tau}{e} \frac{\partial}{\partial \theta} \right]$$

with true anomaly $\tau = l - \theta$ and $w = \sqrt{1 + 2e \cos \tau + e^2}$. Since the drift is unchanged, the pulsation is the same (compare (5)),

$$\omega(l, x) = \frac{n(1 + e \cos \tau)^2}{(1 - e^2)^{3/2}}.$$

The normal maximized Hamiltonian on $\mathbf{S}^1 \times T^*X$ is

$$H(l, x, p) = \frac{\omega}{2} H_1^2(l, x, p),$$

where $H_1 = \langle p, F_1(l, x) \rangle$. Define the averaged Hamiltonian as in (6). It is remarkable that the averaged flow remains Riemannian.

Proposition 4. *The averaged Hamiltonian is*

$$H(x, p) = \frac{p_r^2}{2} + \frac{c^2}{2r^2} \left[\frac{(1 - \sin^2 \varphi)^2}{\sin^2 \varphi (2 - \sin^2 \varphi)^2} p_\theta^2 + p_\varphi^2 \right]$$

with $r = (2/5)n^{5/6}$ and $c = 2/5$.

As in the case of two controls, the flow is Liouville integrable and the whole analysis can be restricted to \mathbf{S}^2 . The metric induced on the sphere is

$$XR(X)d\theta^2 + d\varphi^2, \quad R(X) = \frac{1}{4} \left[1 + \frac{2}{1-X} + \frac{1}{(1-X)^2} \right], \quad X = \sin^2 \varphi. \quad (9)$$

There is now an equatorial singularity at $\varphi = \pi/2$. It is an order two pole at $X = 1$ of the rational fraction R . (Compare with $R = 1/(1 - X)$ in the previous section.)

Let $H_2 = (1/2)[(XR(X))^{-1}p_\theta^2 + p_\varphi^2]$. On the level $H_2 = 1/2$, the quadrature on φ is

$$Y^2 = 4(1 - X)[X(2 - X)^2 - 4p_\theta^2(1 - X)^2], \quad Y = \dot{X}(2-X). \quad (10)$$

The underlying curve is of genus one.⁴ It is parameterized by a doubly periodic Weierstraß function,

$$X = 1 - \frac{1}{\wp(z) - 1/3}, \quad \frac{dt}{dz} = 1 + \frac{1}{\wp(z) - 1/3}, \quad (11)$$

whose invariants reflect the dependence on p_θ ,

$$g_2 = \frac{16}{3} + 16p_\theta^2, \quad g_3 = \frac{64}{27} - \frac{16}{3}p_\theta^2. \quad (12)$$

Without loss of generality, we restrict again the computation to $\theta_0 = 0$ and $\varphi_0 = \pi/2$. With the initial condition at singularity, p_θ^2 is unbounded in contrast to the two-control case. Analyzing roots of the degree three polynomial $4\xi^3 - g_2\xi - g_3$ associated with Weierstraß function, one sees that the parameterization has to be restricted to the unbounded component of the cubic to ensure $X \in [0, 1]$. Hence z belongs to \mathbf{R} .

Proposition 5. *The transcendence for two bodies and one (tangential) control is elliptic. One has*

$$\sin^2 \varphi = \frac{\wp(z) - 4/3}{\wp(z) - 1/3}, \quad z \in \mathbf{R},$$

$$t = \frac{1}{\wp'(a)} \left[\ln \frac{\sigma(z-a)}{\sigma(z+a)} \right]_0^z + \left(1 + \frac{2\zeta(a)}{\wp'(a)} \right) z,$$

$$\theta = 2p_\theta \left[\frac{1}{\wp'(b)} \ln \frac{\sigma(z-b)}{\sigma(z+b)} - \frac{1}{\wp'(c)} \ln \frac{\sigma(z-c)}{\sigma(z+c)} \right]_0^z + 4p_\theta \left(\frac{\zeta(b)}{\wp'(b)} - \frac{\zeta(c)}{\wp'(c)} \right) z,$$

with $\wp(a) = 1/3$, $\wp(b) = 4/3$, $\wp(c) = -2/3$, and invariants (12).

⁴ Excluding the degenerate case $p_\theta = 0$ associated with meridians.

Proof. The quadrature on θ is

$$\frac{d\theta}{dz} = 2p_\theta \left(\frac{1}{\wp(z) - 4/3} - \frac{1}{\wp(z) + 2/3} \right).$$

It is similar to the quadrature (11) on t . Introducing Weierstraß ζ and σ functions, $\wp = -\zeta'$, $\zeta = \sigma'/\sigma$, one has

$$\int \frac{\wp'(a)dz}{\wp(z) - \wp(a)} = 2\zeta(a)z + \ln \frac{\sigma(z-a)}{\sigma(z+a)},$$

whence the result. □

The family of genus one complex curves (10) are all homeomorphic to the torus. The topological classification of extremals is then trivial. We recall standard facts on the moduli space of elliptic curves [18] so as to refine the classification up to conformal equivalence.

Let L be a lattice in the complex plane with basis (l_1, l_2) (complex numbers linearly independent over \mathbf{R}^2). A pair (l'_1, l'_2) defines another basis if only if

$$\begin{aligned} l'_1 &= al_1 + bl_2, \\ l'_2 &= cl_1 + dl_2, \end{aligned}$$

for some matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}(2, \mathbf{Z}).$$

Two tori \mathbf{C}/L , \mathbf{C}/L' are conformal if and only if there is some $\mu \in \mathbf{C}^*$ such that $L' = \mu L$. Let (l_1, l_2) and (l'_1, l'_2) be bases of L and L' , respectively. We can assume that $\tau = l_2/l_1$ and $\tau' = l'_2/l'_1$ belong to Poincaré upper half-plane, \mathbf{H} . From the previous remarks, L and L' are conformal if and only if there is a quadruple (a, b, c, d) such that

$$\tau' = \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \in \mathbf{Z}, \quad ad - bc = 1. \tag{13}$$

Such particular *Möbius transformations* are automorphisms of \mathbf{H} . The induced group morphism between $\text{SL}(2, \mathbf{Z})$ and $\text{Aut}(\mathbf{H})$ has kernel $\pm \text{id}$. Transformations (13) are then identified with the Fuchsian *modular group* $\text{PSL}(2, \mathbf{Z}) = \text{SL}(2, \mathbf{Z})/\pm \text{id}$. Then $\mathbf{H}/\text{PSL}(2, \mathbf{Z})$ is the moduli space for congruences of conformal tori. One eventually defines the *modular function* [18]

$$j(\tau) = \frac{g_2^3}{\Delta}.$$

It is a bijection from $\mathbf{H}/\text{PSL}(2, \mathbf{Z})$ onto \mathbf{C} .

In our case, to each p_θ is associated a real rectangular lattice. Using (12), one can define

$$j(p_\theta) = \frac{16(1 + 3p_\theta^2)^3}{27p_\theta^2(8 + 13p_\theta^2 + 16p_\theta^4)} \tag{14}$$

and obtain the following classification of extremals.

Proposition 6. *There are not more than three conformal φ -curves.*

Proof. Function (14) has exactly two critical points, so $j(p_\theta) = \text{constant}$ has at most three distinct solutions (not taking into account symmetric solutions, $\pm p_\theta$). \square

To estimate the conjugate locus at singularity, we use the following local model. Set $x = \pi/2 - \varphi$, $y = \theta$. The metric (9) is locally approximated by

$$dx^2 + \frac{dy^2}{x^{2p}}. \tag{15}$$

In the case of one (tangential) control, $p = 2$.

Proposition 7 ([7]). *The conjugate locus at the origin of (15) is $y = \pm C_p x^{p+1}$ minus the origin itself. As $p \rightarrow \infty$, $C_p \sim \boxed{2 \sqrt{2} / p}$.*

As a result, the conjugate locus of the metric on S^2 has an order 3 contact with the equatorial singularity. Because of the symmetry $p_\theta \rightarrow -p_\theta$, this defines two meridional cusps of the conjugate locus at $\varphi_0 = \pi/2$. (See Fig. 2.) The result of section 2 can be extended to this singular setting.

Theorem 5 ([7]). *If $\Delta\theta$ is strictly decreasing convex, the conjugate locus has four (possibly degenerate) cusps, all meridional for equatorial points, two horizontal and two meridional otherwise.*

The verification on $\Delta\theta$ is intricate but can again be made. The following estimates are computed in [7],

$$\Delta\theta \sim_0 2\pi\left(1 - \frac{3\sqrt{2}}{4}p_\theta + \frac{35\sqrt{2}}{128}p_\theta^3\right), \quad \Delta\theta \sim_\infty \frac{4}{3}(2 - \sqrt{2})K(3 - 2\sqrt{2})p_\theta^{-3/2},$$

where $K(k)$ is the complete Jacobi integral of first kind and modulus k . The previous structure result on the cut locus is also extended to include the two-body one-control case.

Theorem 6 ([7]). *If $\Delta\theta$ is strictly decreasing, the cut locus of a point on the sphere is the equator minus the point itself for equatorial points, a closed antipodal subarc otherwise.*

4 Three bodies, two controls

In contrast with sections 2 and 3, we keep the original constraint on the control, and consider the final time minimization of

$$\ddot{q} + \nabla V_\mu(q) + 2iq = \varepsilon u, \quad |u| \leq 1.$$

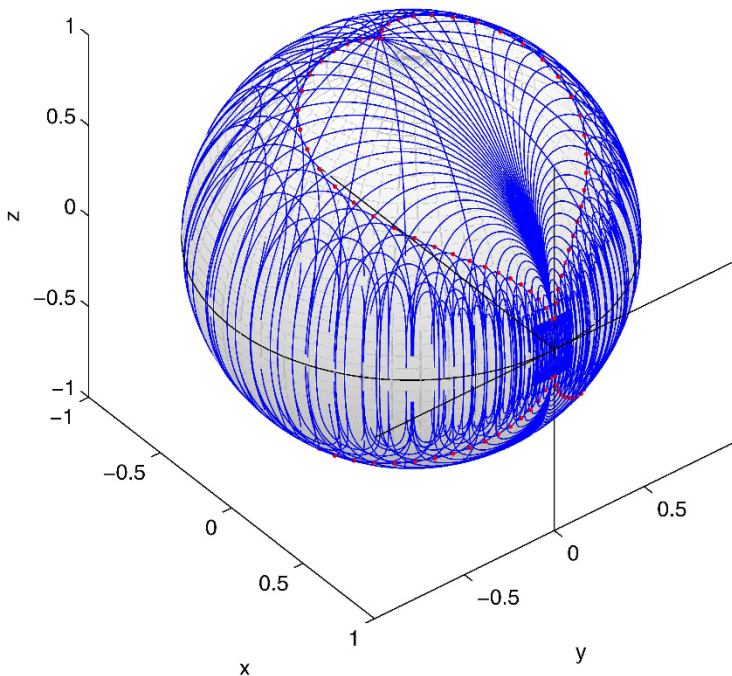


Fig. 2. Conjugate locus, two bodies and one (tangential) control. The double-heart locus (in red) is the envelope of geodesics (in blue) emanating from the initial point. It has four meridional cusps (two of them generated by order 3 contacts at origin). The cut locus (in black) is the whole equator minus the origin.

See [9] for preliminary computations on the L^2 -relaxation of the problem. Available results on controlled three-body problems are mostly numerical. They usually deal with refined models taking into account three-dimensional effects, perturbations, and rely on direct optimization methods. (See, *e.g.*, [3].)

The position vector q belongs to the complex plane with two punctures at $-\mu$ and $1 - \mu$, denoted Q_μ . The state space X_μ is the tangent space TQ_μ in (rotating) cartesian coordinates (q, \dot{q}) . It is the cotangent space T^*Q_μ in (q, p) variables, see (1). In both cases, $X_\mu \simeq Q_\mu \times \mathbf{R}^2$ is a trivial bundle. In cartesian coordinates,

$$\dot{x} = F_0(x) + \varepsilon(u_1 F_1(x) + u_2 F_2(x)), \quad |u| \leq 1.$$

There,

$$F_0(x) = \dot{q} \frac{\partial}{\partial q} - (\nabla V_\mu(q) + 2iq) \frac{\partial}{\partial \dot{q}}, \quad F_1(x) = \frac{\partial}{\partial \dot{q}_1}, \quad F_2(x) = \frac{\partial}{\partial \dot{q}_2}.$$

The maximized normal Hamiltonian⁵ is

$$H = -1 + H_0 + \varepsilon \sqrt{H_1^2 + H_2^2}, \quad H_i = \langle p, F_i(x) \rangle, \quad i = 0, \dots, 2.$$

Extremals are classified according to the order of their contact with the switching surface $\Sigma = \{H_1 = H_2 = 0\}$. (See [12].)

Proposition 8. *Contacts with Σ are of order one and define isolated π -singularities.*⁶

Proof. The distribution $\{F_1, F_2\}$ being involutive, the switching function $\psi = (H_1, H_2)$ is \mathcal{C}^1 ,

$$\dot{\psi}_1 = \{H_0, H_1\} - u_1 \{H_1, H_2\}, \quad \dot{\psi}_2 = \{H_0, H_2\} + u_2 \{H_1, H_2\}.$$

The bracket $\{H_1, H_2\}$ vanishes on Σ . The drift comes from a second order mechanical system, so $\{F_1, F_2, [F_0, F_1], [F_0, F_2]\}$ has full rank. Then $\dot{\psi} \neq 0$ on Σ and contacts are of order one. By Pontryagin maximization condition, $u = \psi/|\psi|$, so u is changed to $-u$ whenever ψ vanishes. \square

As Σ is of codimension two in T^*X_μ , we can neglect these finitely many π -singularities for the numerical computation and restrict to smooth extremals not crossing the switching surface.

For the minimum time problem, the exponential mapping associated with order zero extremals is defined on a neighbourhood of the origin in $\mathbf{R} \times H(x_0, \cdot)^{-1}(0)$,

$$\exp_{x_0} : (t, p_0) \mapsto \Pi \circ \exp t \overrightarrow{H}(x_0, p_0) = x(t, x_0, p_0), \quad \Pi : T^*X_\mu \rightarrow X_\mu.$$

Given a target x_f , the problem is to find a zero of the shooting equation

$$\exp_{x_0}(t_f, p_0) = x_f.$$

The two-body problem is embedded into the three-body one thanks to parameter μ . This paves the way for using continuation methods between two and three-body control problems. (See also [15] for such an approach in the two-body case.) Rather than information on the adjoint, the knowledge of the Kepler minimum time from [12] turns out to be critical to initialize the continuation. Our target for numerical computation is an equilibrium point of the uncontrolled problem or *Lagrange point* [26]. Such points where the influences of the two primaries compensate each other are appropriate targets for the continuation. Here we use the L_2 Lagrange point. It is equal to

⁵ From now on, p denotes the adjoint to x .

⁶ Instantaneous rotations of angle π of the control.

the second primary when $\mu = 0$. Lagrange points are extensively studied in celestial mechanics and mission design [19]. A second continuation on ε is also used to reach low bounds on the control, see results in Fig. 3. Hill regions are projections on the q -space of level sets of the Jacobi integral. In the controlled case, they vary dynamically along the trajectory (see Fig. 4),

$$R_\mu(t) = \{\xi \in Q_\mu \mid J_\mu(q(t), \dot{q}(t)) - V_\mu(\xi) \geq 0\}.$$

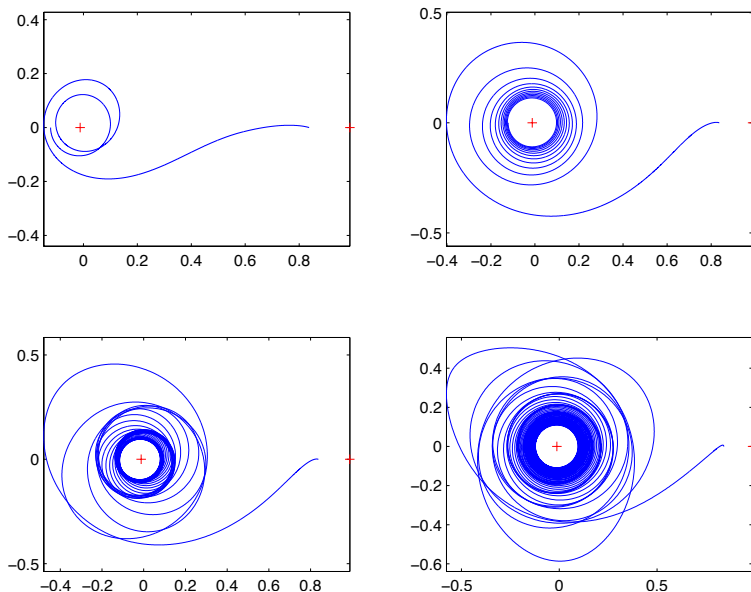


Fig. 3. Three bodies, two controls. Minimum time trajectories from the geostationary orbit to Lagrange L_2 point in the Earth-Moon system ($\mu \simeq 0.0121$). Successively, $\varepsilon = 2.440, 0.2440, 0.1220$ and 0.04148 .

A normal extremal is *regular* if it verifies the strong Legendre condition that there exists some positive α such that, everywhere on $[0, t_f]$, $\partial^2 H / \partial u^2 \leq -\alpha I$ along the extremal.

Lemma 1. *Order zero extremals of the minimum time three-body problem with two controls are regular.*

Proof. Along an order zero extremal, $u \in \mathbf{S}^1$. In any chart,

$$\frac{\partial^2 H}{\partial u^2} = -\varepsilon \sqrt{H_1^2 + H_2^2} = -\varepsilon |\psi|.$$

The absence of π -singularity implies the strong Legendre condition as $|\psi|$ is then smooth and bounded below by some positive constant. \square

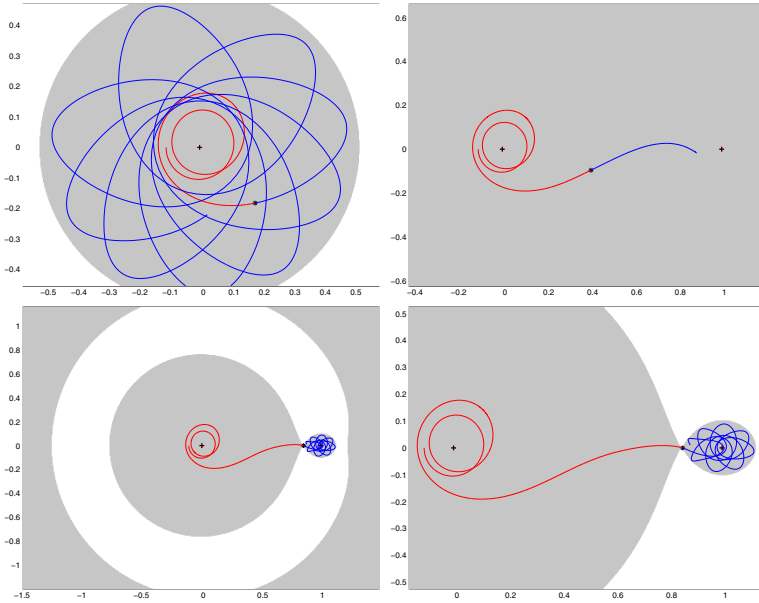


Fig. 4. Dynamics of the Hill regions, $\varepsilon = 2.440$. The controlled trajectory (in red) is plotted up to three different times and prolonged by the oscillating uncontrolled trajectory (in blue). During a first phase, energy J_μ is increased so as to include the L_2 target. The second phase is close to the motion of the system towards projection in the q -space of the Lagrange point. The last two graphs are identical (the rightmost one has a finer scale) and illustrate instability of the L_2 point after the target is reached.

As in the Riemannian case (without singularities), regular extremals are locally time minimizing for short times [1]. To investigate further local optimality, one generalizes Jacobi theory to the optimal control setting. Define again conjugate points as critical values of the exponential mapping. The following technical condition is sufficient to avoid degenerate situations on the kernel of the second variation of the problem (see [24]). Let

$$E_{x_0} : (t_f, u) \mapsto x(t_f, x_0, u)$$

be the *endpoint mapping*. It is defined on a neighbourhood of the reference pair (t_f, u) in $\mathbf{R} \times L^\infty([0, t_f], \mathbf{S}^1)$. We assume that, for any subinterval $[t_1, t_2]$ of $[0, t_f]$, the partial derivative $\partial E_{x(t_1)} / \partial u(t_2 - t_1, u|_{[t_1, t_2]})$ has corank one.

Theorem 7 ([1, 24]). *Under the corank one assumption, the trajectory associated with a regular extremal is \mathcal{C}^0 -locally time minimizing up to the first conjugate point. Past this point, the control is not even L^∞ -locally minimizing.*

Local optimality of every extremal is verified by a conjugate point test. (See Fig. 5). The practical computation of conjugate points is done by rank evaluation on Jacobi fields [11]. The concept of conjugate point is extended by

the notion of *focal point* [Ibid.] to encompass the case of submanifold targets. Such an example for a lunar orbit target is provided Fig. 6.

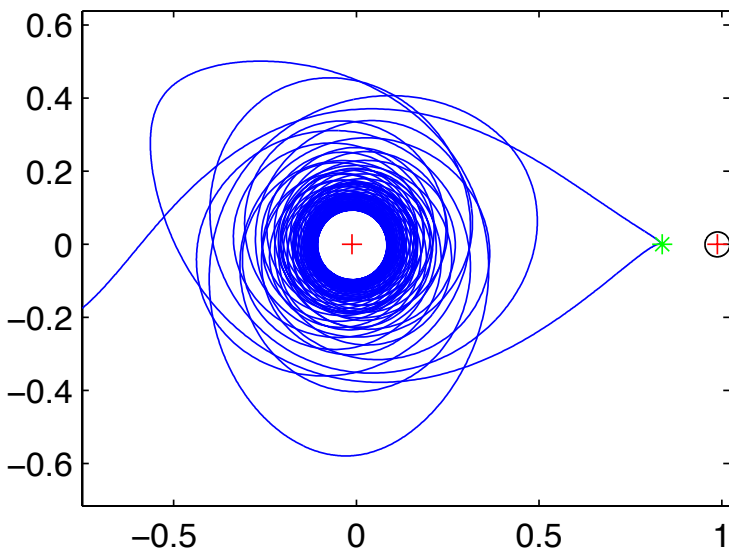


Fig. 5. Conjugate point computation, $\varepsilon = 0.04148$. The reference trajectory is prolonged up to the first conjugate point, beyond the L_2 target. Local optimality up to the target is guaranteed. The cuspidal point of first kind observed is generically due to the condition $\dot{q}_f = 0$.

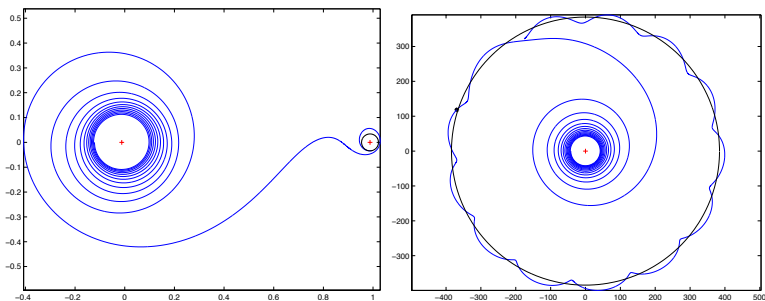


Fig. 6. Focal point computation, $\varepsilon = 0.2440$. The target is a lunar orbit, and the focal point test ensures local optimality of the trajectory. The leftmost frame is the rotating frame, the rightmost one is fixed.

Whatever the target, the value function $\varepsilon \mapsto t_f(\varepsilon)$ of the minimum time problem is decreasing: The smaller ε , the larger the transfer time. This is

contradicted by results portrayed Fig. 7. We infer that the first extremal is locally but not globally minimizing. When decreasing the bound on the control ε from 0.2221 to 0.2196, one revolution around the first primary has to be added before escape towards the second body is obtained. There lies global analysis of the problem, in the interplay between the two small parameters μ , ε . This leaves open the question of global optimality.

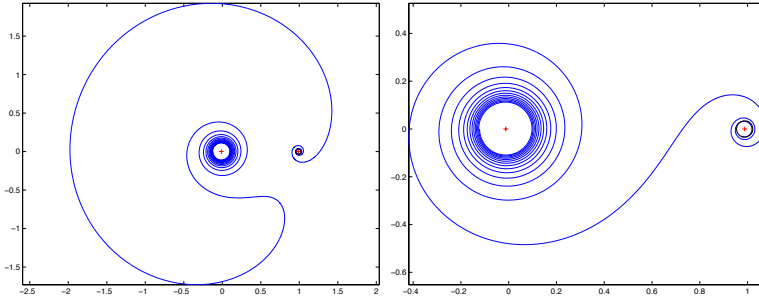


Fig. 7. Lunar target, $\varepsilon = 0.2221$ and 0.2196 . Focal point tests ensure local optimality in both cases. However, $t_f \simeq 17.8$ versus $t_f \simeq 10.8$ in the second one. The first extremal is a local but not a global minimizer. The difference in strategies is apparent as one extra revolution around the Earth is added in the second case before reaching the lunar orbit target.

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